Technical Report No. 7

SACLANT ASW RESEARCH CENTER

A NOTE ON CONVERGENT ZONES

by

R. R. GOODMAN

L. R. B. DUYKERS

2 February 1962

OPERATED BY SIRIMAR
VIALE SAN BARTOLOMEO, 92
LA SPEZIA (ITALIA)

THE SUPREME ALLIED COMMANDER ATLANTIC
ANTISUBMARINE WARFARE RESEARCH CENTER
LA SPEZIA, ITALY

2 February 1962

To: DISTRIBUTION LIST

Subj: SACLANT Anti-Submarine Warfare Research Center Technical Report

Encl: (1) Subject Report

1. Enclosure (1) is forwarded for information and retention. A limited number of additional copies are available at the Center if needed.

2. Addressees are requested to inform the Center of any changes desired in the number of copies of future technical reports to be forwarded.

J.O. PHILLIPS
Captain, U.S. Navy
SACLANT Deputy

JOHN M. IDE
Scientific Director

BASIC DISTRIBUTION LIST:

SACLANT	(10)	MOD, Netherlands	(10)
MOD, Canada	(10)	MOD, Norway	(10)
MOD, Denmark	(10)	MOD, U.K.	(14)
MOD, France	(10)	SCIENTIFIC ADV NATO	(5)
MOD, Germany	(10)	SACEUR	(7)
MOD, Italy (Navy General Staff)	(10)		

Additional Distribution as authorized by SACLANT:

Director, Mutual Weapons Development Team, Paris, France (2)
President, Pennsylvania State University (1)
Secretary of Defense (Director of Defense Research and Engineering) (2)
Chief of Naval Operations (2)
Office of Naval Research (Code 400) (1)
Bureau of Ships (Code 600) (1)
Bureau of Weapons (Code CU) (1)
Operations Evaluation Group (1)
Weapons Systems Evaluation Group (1)
Commander, Operational Test and Evaluation Force (1)
National Academy of Sciences (Committee on Undersea Warfare) (1)
Naval Research Laboratory, Washington, D. C. (1)
Naval Ordnance Laboratory, White Oak, Maryland (1)
U. S. Navy Underwater Sound Laboratory, New London, Connecticut (1)
Ordnance Research Laboratory (The Pennsylvania State University) (1)
Naval Ordnance Test Station, China Lake, California (1)
Hudson Laboratories (Columbia University) (1)
Woods Hole Oceanographic Institution, Woods Hole, Mass. (1)
Scripps Oceanographic Institution, San Diego, California (1)
Naval Electronics Laboratory, San Diego, California (1)
Naval Air Development Center, Johnsville, Pennsylvania (1)
Applied Physics Laboratory, (University of Washington) (1)

A NOTE ON CONVERGENT ZONES

by

R. R. GOODMAN

L. R. B. DUYKERS

Technical Report No. 7
Date: 2 February 1962

ABSTRACT

An exact solution for ray paths near a sound channel with parabolic velocity profile is given for rays crossing the sound channel axis at small angles. The convergent zone for such rays for a source on axis is compared with the numerically computed results of M. A. Pederson, J. Acoust. Soc. Am., 33, 465, (1961). Two simple straight line velocity gradient approximations are compared with the parabolic case for the purpose of observing their accuracy. Both cases show surprisingly good agreement over a range of choices for the gradients.

I. INTRODUCTION

The problem of computing ray paths for sound rays under actual oceanographic conditions is known to be difficult due to the complicated depth profiles generally found. This is especially true under general summer conditions in which there exists a strong velocity gradient just below the surface channel[1]. The usual approach to ray tracing is one in which the velocity profile is approximated by straight line segments used over various depths. Such an approximation will obviously lead to errors in both the ray paths and in the predicted convergent zones[2]. It is the purpose of this report to present a simple solution given in Section II, for the ray paths and convergent zones for sound velocity profiles exhibiting minima. The solution given is valid for rays leaving the source at small angles from the channel axis. For convenience this paper is limited to sound sources on the sound channel axis. The extension of the source to off axis position is straightforward as long as the small angle requirement mentioned above is met. Since the constant gradient approach to velocity profiles is so useful in many problems in Section III two very simple constant gradient examples are fitted to the parabolic profile and their convergent zones are compared with the parabolic results. The possible use of the parabolic result is discussed in Section IV, as well as the merit of the straight line method. The possibility of considering the types of profiles by the above method is mentioned.

---

[1] C. B. Officer. Introduction to the Theory of Sound Transmission, p. 150. McGraw-Hill Book Co. New York. (1958).

[2] Melvin A. Pederson. J. Acoust. Soc. Am. 33, 465 (1961).

II. THE RAY PATH FOR A PARABOLIC VELOCITY PROFILE

There exists a very simple solution for the ray equation for small angles if the velocity has a depth dependence given by

$$C(Z) = C_o (1 + \alpha^2 Z^2) . \tag{1}$$

Where the axis $Z = O$ (corresponding to what shall be called the X axis) is taken at the minimum. Typical numbers given by Pederson[1] indicate that for values of reasonably close to the axis ($Z = O$) the error in approximating the velocity profile by this form is less than one per cent. This is seen from the equation given by him on page 471 of his article. From Snell's Law

$$\frac{1}{\sqrt{1 + Z'^2}\; C(Z)} = \frac{\cos \theta_o}{C_o} \tag{2}$$

where θ_o is the angle of emergence of a ray from the source and $Z' = \frac{dZ}{dX}$, it may be seen that if $C(Z)$ only varies slightly from C_o then Z'^2 is also small. The condition on $C(Z)$ is certainly the oceanographic case. If θ_o is small, the substitution of Eq. (1), expansion of Eq. (2), and omission of higher order terms leads to the differential equation

$$\frac{1}{2} Z'^2 + \alpha^2 Z^2 = \frac{1}{2} \theta_o^2 . \tag{3}$$

---

[1] M. A. Pederson. J. Acoust. Soc. Am. <u>33</u>, 465 (1961).

This has the same form as the Hamiltonian for the harmonic oscillator in classical mechanics, the solution of which is

$$Z = \frac{\theta_o}{\sqrt{2}\,\alpha} \sin \sqrt{2}\,\alpha X \quad . \tag{4}$$

It is then immediately obvious that all rays, independent of θ_o, cross the axis for the first time at the range

$$R = \frac{\pi}{\sqrt{2}\,\alpha} \tag{5}$$

This is correct up to the order $\theta_o^{\,2}$. From the numbers given in Pederson's article on page 471[1], the predicted convergent zone using Eq. (5) is R = 5,623 yards, which is in very good agreement with Pederson's value, 5,539 yards. The discrepancy is due mostly to the approximation of Pederson's velocity equation by a parabolic function.

---

[1] M.A. Pederson. J. Acoust. Soc. Am. 33, 465 (1961).

III. APPROXIMATIONS TO THE ABOVE RESULTS USING STRAIGHT LINE SEGMENTS

With the above result giving a simple range formula it is of interest to compare with it the results of the straight line segment method of approximating velocity profiles. The two different cases considered are shown in Fig. 1. The first is a line segment of constant velocity joined with a line segment tangent to a point on the parabola. Very simple calculations given in Appendix A show for this case a convergent zone

$$R_1 = \frac{2}{\alpha} \tag{6}$$

which is 90.5% in range of that given by Eq. (5). This is surprisingly good agreement for such a simple approximation. The second approximation considered is that given by two line segments as shown in Fig. 1. The point nearest the axis was taken as the fraction ξ of the height of the second point. The calculated convergent zone as a function of ξ as derived in Appendix A is

$$R_2 = \frac{2\sqrt{2\xi(\xi+2)}}{\alpha(1+\xi)} \quad . \tag{7}$$

This function $R_2(\xi)$ monotonically increases over $o \leq \xi \leq 1$. It rapidly rises to about 90% of the value given by the parabolic solution (Eq.(5)) and increases to about 110% of that value at $\xi = 1.0$. At the value $\xi = 0.61$, R_2 is equal to the parabolic value $\pi/\sqrt{2}\,\alpha$. Thus as long as ξ is greater than 0.4 the agreement with the parabolic convergent zone is very good.

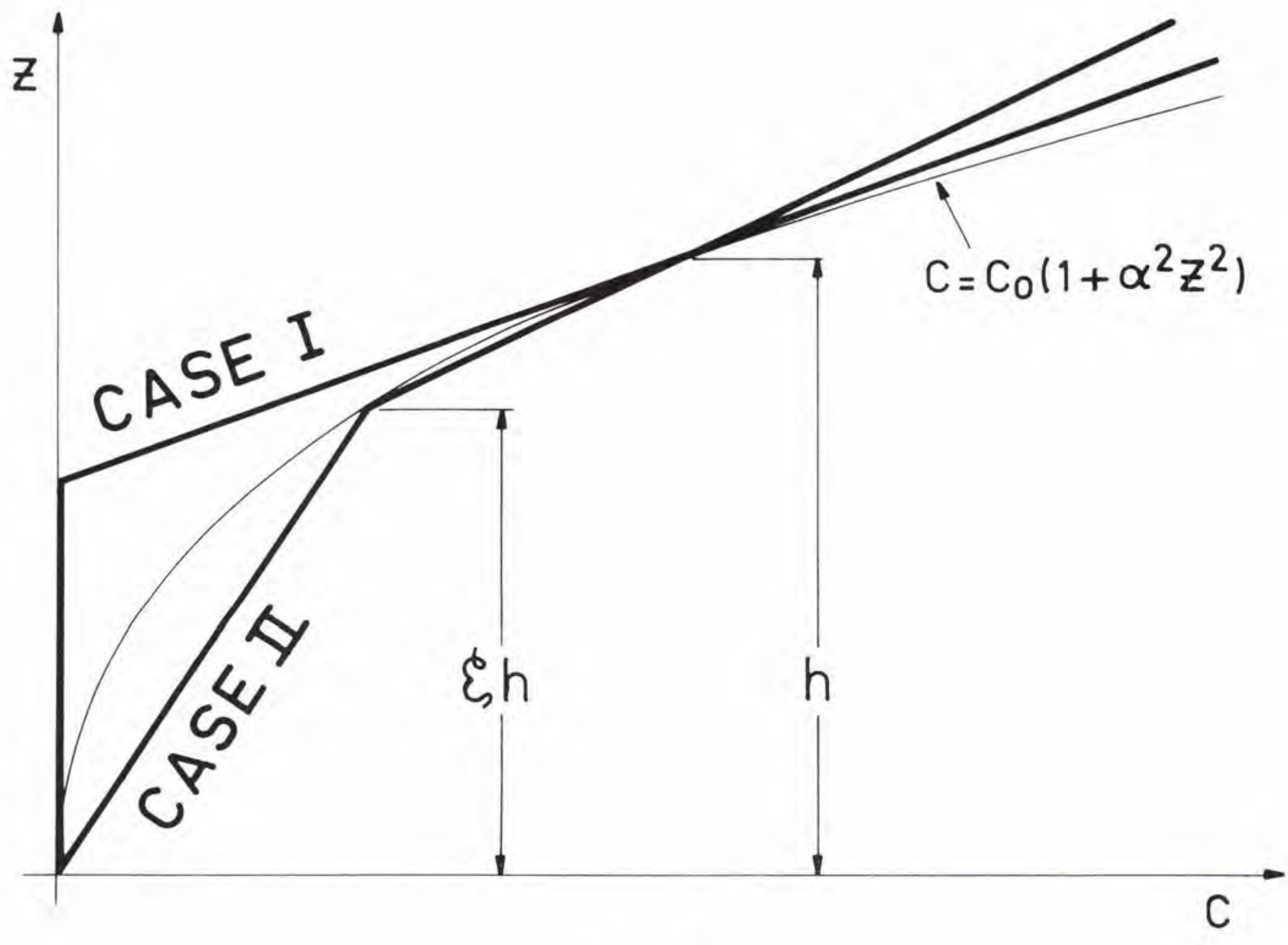

Fig. 1

Velocity Profiles used to Approximate Parabolic

Depth Dependent Curve $C = C_o (1 + \alpha^2 Z^2)$.

IV. DISCUSSION

The small angle approximation has allowed a very simple solution to the ray path. Thus in regions where there exists a minimum in velocity profiles, there exists a region over which this solution may be used. The limitations imposed by actual cases would be due to the actual width of the "parabolic" region in the sound channel. If the deviation from parabolic shape is too great over a small width then in order to include a sufficient range of ray angles the method would have to be extended by connecting other types of gradients to the parabolic one. It should be noted that when considering ray paths due to the sound channel, it is only necessary to consider the velocity profile on one side of the axis since the two sides may be treated separately. Thus much of the actual asymmetry of the profile may be ignored.

The approximation methods used in Section III give remarkably good agreement with the parabolic results of Section II. This agreement makes one more confident that simple models of complicated profiles are still useful for some purposes. Pederson's comment concerning anomalous caustics is not meaningful for the two velocity gradient case. It does become important, however, if one or more gradients are added to the cases considered here. One additional gradient added to case one in Section III produces another convergent zone which was found to wander about drastically, depending on the relative depths of the layers. The same convergent zone given by Eq. (6) is still present. Thus, as Pederson pointed out, caution must be used in interpreting the results.

It is interesting to note that for a velocity versus depth function of the form

$$C(Z) = C_o \left(1 + f(Z)\right) \tag{8}$$

where $f(Z) \ll 1$ for all Z the differential equation

$$\frac{1}{2} Z'^2 + f(Z) = \frac{1}{2} \theta_o^2 + f(Z_o)$$

for rays is obtained from Snell's Law for small angles. Z_o is the depth of the source and θ_o is the angle with the horizontal of the outgoing ray. This equation has the form of a Hamiltonian for arbitrary potentials $f_{(Z)}$. Thus many of the presently used solutions of classified mechanics may be useful in ray tracing problems (for small angles). One of the authors (R. R. Goodman) is currently investigating this possibility.

APPENDIX A

In this section the convergent zones will be calculated for the two gradient cases shown in Fig. 1. The source is considered on the sound channel axis. First the distance at which a ray leaving the source at angle θ_o crosses the axis must be found. Call this distance $R(\theta_o)$. The result for case one is easily found by well known means[1] to be

$$R_1(\theta_o) = \frac{2\,d}{\tan\theta_o} + \frac{2\,C_o}{g_1}\tan\theta_o \tag{A-1}$$

where d is the height of the constant velocity region and g_1 is the velocity gradient above. Since g_1 is the tangent to the slope of the parabola at some point Z = h. By Eq. (1) it is seen that

$$g_1 = 2C_o\alpha^2 h\,. \tag{A-2}$$

The value d is also easily seen to be h/2. In order to calculate the position of the convergent zone it is only necessary to find the value of θ_o such that $\frac{d\,R_1}{d\,\theta_o} = 0$. This value is seen to be

$$\tan\theta_o = \sqrt{\frac{d\,g_1}{C_o}}\quad . \tag{A-3}$$

---

[1] C. B. Officer. Introduction to the Theory of Sound Transmission, Chapter II. McGraw-Hill Book Co. New York. (1958).

When Eqs. (A-2), (A-3) and $d = \frac{h}{2}$ are substituted into (A-1) the convergent range is seen to be

$$R_1 = \frac{2}{\alpha} \quad . \tag{A-4}$$

The second case is calculated along similar lines. The range at which a ray crosses the axis is given by

$$R_2(\theta_o) = \frac{2C_o}{g_1} \tan \theta_o - \frac{2C_1}{g_1} \tan \theta + \frac{2C_1}{g_2} \tan \theta \tag{A-5}$$

where g_1 and g_2 are the two gradients is seen in Fig. 1. C_1 and θ are the values of the velocity and angle of the ray respectively at the depth of intersection of the two line segments. Setting $\frac{dR_2}{d\theta_o} = O$, using the fact that

$$\frac{d\theta}{d\theta_o} = \frac{C_1}{C_o} \quad \frac{\sin \theta_o}{\sin \theta} \quad , \tag{A-6}$$

which follows from Snell's Law, an equation for θ_o is obtained which is

$$\tan \theta_o = \sqrt{\frac{(C_1^{\ 2} - C_o^{\ 2})\, g_2}{C_o^{\ 2} \left[g_2^{\ 2} - (g_2 - g_1)^2 \right]}} \quad . \tag{A-7}$$

Since the points h and ξh on the velocity profile determine g_1, g_2 and C_1, it is easily shown that

$$g_1 = C_o \alpha^2 \xi h$$

$$g_2 = C_o \alpha^2 (1 + \xi)h \qquad \text{(A-8)}$$

$$C_1 = C_o (1 + \alpha^2 \xi^2 h^2) .$$

Substituting Eqs. (A-7), (A-8) and Snell's Law into (A-5), the range of the convergent zone is found to be

$$R_2 = \frac{2 \sqrt{2 \xi (\xi + 2)}}{\alpha (1 + \xi)} . \qquad \text{(A-9)}$$