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THE PROPAGATION AND RECEPTION OF TRANSIENT ELECTROMAGNETIC SIGNALS IN THE PRESENCE OF A CONDUCTING HALF-SPACE

by

P.E. MIJNARENDS

May, 1961

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ABSTRACT

In this paper a theoretical treatment is given of the short-distance propagation and reception of a step-function in the presence of a conducting half-space. The step-function is generated by a horizontal electric dipole situated in the conducting half-space. First the waveforms of the electric and magnetic fields are computed for points in the conductor, making use of Laplace-transform theory. Since the vertical electric field is discontinuous at the interface its waveform in points just above the conductor is derived separately. It appears that for horizontal distances larger than three times the antenna depth the attenuation of this vertical electric transient with distance is equal to that in the harmonic case, viz. proportional to the inverse square of distance. Moreover, smearing out of the signal as it is propagated in the horizontal direction does not occur, contrary to that of the other components. The waveforms of the vertical electric field and the component of the horizontal magnetic field parallel to the antenna are plotted.

The second part deals with the reception of transients by detectors (e.g. coils) having a frequency characteristic proportional to some power of frequency. It is shown that a rising frequency characteristic (differentiator) may cause an extra attenuation of the transient as a whole with horizontal distance while a falling characteristic (integrator) reduces the attenuation. The response of a system with a $\omega^{\frac{1}{2}}$ characteristic is treated extensively. Finally an unproven hypothesis is given, establishing a simple relation between the slope of the frequency characteristic of the detector and the extra attenuation suffered by the transient.

INTRODUCTION

In the search for improved methods for detecting and classifying submarines the possibility of using electromagnetic waves for this purpose is being investigated. In this there are involved problems concerning the transmission, propagation, reflection and reception of electromagnetic signals. These signals need not necessarily be sinusoidal in their time dependence; also signals of a different shape, here generally called "transients", are possible and may sometimes even be preferred. Using this definition, every practically realizable signal is in fact a transient since a pure sinusoidal wave has an infinite length. In this paper the propagation and reception of a special type of transient, the step-function, in the presence of a conducting half space, will be investigated theoretically.

The problem of the propagation of electromagnetic signals over a conducting earth was first treated by Sommerfeld[1)] in 1909. He gave the fundamental integrals which describe the behaviour of the electromagnetic field, assuming a harmonic time dependence. Since these integrals do not lend themselves easily to numerical computations much work has been done by various investigators[2,3,4)] in order to find approximate expressions which are more suited for numerical purposes. These formulas are usually valid for restricted ranges of the various parameters and variables involved, such as conductivity of the earth, height above or below the surface and horizontal range. For the case of a horizontal electric dipole situated in the conducting half space a fairly complete treatment of this subject has been given lately by Baños and Wesley[5)], who in their paper also give an extensive list of references to the earlier literature. Though they also give solutions for other ranges their main interest is directed to the near field, which is the case of interest in detection of and short-range communications with submerged submarines.

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1) A. Sommerfeld, Ann. Physik 28 (1909), 665-737
2) B. van der Pol, Z. Hochfrequenz. Tech. 37 (1931), 152-157
3) H. Ott, Ann. Physik 41 (1942), 443-466; Ann. Physik 43 (1943), 393-404
4) S.O. Rice, Bell System Tech. J. 16 (1937), 101-109
5) A. Baños jnr., J.P. Wesley. The Horizontal Electric Dipole in a Conducting Half Space. Vols I and II. (Scripps Institution of Oceanography, SIO References 53-33 and 54-31). This paper contains a fairly complete list of papers dealing with the subject.

All these studies have in common the fact that they assume a harmonic time dependence of the antenna current. In practice, however, transmitted signals may have different shapes. Due to the large dispersion which electromagnetic waves suffer in a highly conducting medium like sea-water the signal shape will be, in that case, greatly altered during the propagation. This problem has more recently begun to receive much interest, especially because of its importance in geophysical prospecting. Wait, in one of his papers[6), has given an extensive survey of the existing literature, both of Western and Russian origin, on this subject. In the same paper he treats the propagation of transients generated by a horizontal electric dipole, limiting himself, however, to the description of the behaviour of the electric field in the case of excitation of the antenna with a delta-pulse. As mentioned above, the paper here will be concerned with the propagation of step-functions, this being a basic type of signal from which other signal shapes (square pulse, repetitive square wave) can be derived by superposition. In addition to the electric field components the magnetic field components also will be derived for points of observation in the conducting medium and at the interface. Since all components are continuous at the interface, except the z component of the electric field, a separate expression has to be derived for this component in points lying in the non-conducting medium just above the interface.

In addition to the effects of propagation the signal shape may also be influenced by the frequency response of the receiving equipment. This will, in general, occur when the response of the receiver is not flat within the frequency range occupied by the transient. A simple example of this is the reception of magnetic signals with the aid of a coil. As a result an important quantity like the range dependence of the signal amplitude, as obtained from a recorder connected to the output of the receiver, may differ greatly from that expected on the basis of computations which only take into account the propagation of the transient and do not consider the influence of the receiver. It is felt that this important fact is often not fully realized.

This paper is divided into two major sections. Section 1 will be concerned with the derivation of the responses of both the electric and the magnetic field to a step-function excitation. In Section 2 the influence on the signal shape of a few simple receiver characteristics will be examined. Some mathematical details will be found in the Appendices.

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6) J. R. Wait. Appl. Sci. Research B8 (1960), 213-253.

Section 1. Propagation of Transient Signals

1.1 Theory

In the following a cartesian coordinate system will be used with its origin lying in the interface separating the two media. The conducting medium (sea water in the case of interest) with conductivity σ and dielectric constant ε_1, will occupy the half-space $z > 0$, while the half space $z < 0$ is non-conducting and has a dielectric constant ε_0. The magnetic permeability μ_0 is the same in both media. An infinitesimal electric dipole of length $d\ell$ is situated on the positive Z-axis at the point (o,o,h). Its direction is parallel to the X-axis, as indicated in Figure 1. The point of observation, P, will be described by its coordinates (x,y,z). All quantities bearing relation to the non-conducting medium will be denoted by the subscript zero, those related to the conducting medium will bear the subscript 1. The MKS unit system will be used.

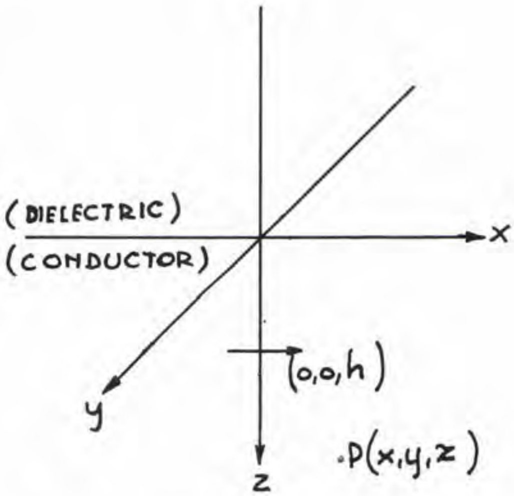

Fig. 1. Coordinate system with dipole in conducting half space.

As shown by Sommerfeld[7] the fields may be derived in the case of a horizontal dipole from a Hertz vector $\underline{\Pi}$ possessing both an x and a z component, given for the non-conducting medium ($z < 0$) by

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7) A. Sommerfeld, Vorlesungen über Theoretische Physik, Band VI. (Dieterich'sche Verlagsbuchhandlung, Wiesbaden, Germany, 1947). pp. 260 ff.

$$\Pi_{0x} = \frac{I\,dl}{2\pi i \varepsilon_0 \omega} \int_0^\infty \frac{e^{u_0 z - u_1 h}}{u_0 + u_1} J_0(\lambda \rho)\, \lambda \, d\lambda \qquad \text{Eq.(1)}$$

$$\Pi_{0y} = 0$$

$$\Pi_{0z} = -\frac{I\,dl}{2\pi i \varepsilon_0 \omega} \frac{\partial}{\partial x} \int_0^\infty \frac{(u_1 - u_0)\, e^{u_0 z - u_1 h}}{\gamma_0^2 u_1 + \gamma_1^2 u_0} J_0(\lambda \rho)\, \lambda \, d\lambda \qquad \text{Eq.(2)}$$

and for the conducting medium $(z > 0)$ by

$$\Pi_{1x} = \frac{I\,dl}{4\pi\sigma} \left[\frac{e^{-\gamma_1 R_1}}{R_1} - \frac{e^{-\gamma_1 R_0}}{R_0} + 2 \int_0^\infty \frac{e^{-u_1(z+h)}}{u_0 + u_1} J_0(\lambda\rho)\, \lambda \, d\lambda \right] \qquad \text{Eq.(3)}$$

$$\Pi_{1y} = 0$$

$$\Pi_{1z} = -\frac{I\,dl}{2\pi\sigma} \frac{\partial}{\partial x} \int_0^\infty \frac{(u_1 - u_0)\, e^{-u_1(z+h)}}{\gamma_0^2 u_1 + \gamma_1^2 u_0} J_0(\lambda\rho)\, \lambda \, d\lambda \qquad \text{Eq.(4)}$$

where

$$\rho^2 = x^2 + y^2 \qquad \gamma_0^2 = -\varepsilon_0 \mu_0 \omega^2$$

$$R_0^2 = \rho^2 + (z+h)^2 \qquad \gamma_1^2 = i\sigma\mu_0\omega - \varepsilon_1 \mu_0 \omega^2 \approx i\sigma\mu_0\omega$$

$$R_1^2 = \rho^2 + (z-h)^2 \qquad u_0^2 = \lambda^2 + \gamma_0^2$$

$$u_1^2 = \lambda^2 + \gamma_1^2$$

I denotes the current strength, λ is a variable of integration and ω represents the angular frequency.

In Equations (1) to (4) displacement currents in the conducting medium have been neglected which is equivalent to neglecting $\frac{\omega \varepsilon_1}{\sigma}$ compared with unity. In the case of sea water this is permitted for frequencies below 10 Mc/s. It can readily be shown that the Hertz vectors, as given above, satisfy the boundary conditions at $z = 0$:

$$\gamma_0^2 \underline{\Pi}_0 = \gamma_1^2 \underline{\Pi}_1$$
$$\gamma_0^2 \frac{\partial \Pi_{0x}}{\partial z} = \gamma_1^2 \frac{\partial \Pi_{1x}}{\partial z} \qquad \text{Eq.(5)}$$
$$\text{div}\, \underline{\Pi}_0 = \text{div}\, \underline{\Pi}_1$$

The electric and magnetic fields are derived from the Hertz vectors according to

$$\underline{E}_0 = -\gamma_0^2 \underline{\Pi}_0 + \text{grad. div}\, \underline{\Pi}_0 \qquad \text{Eq.(6a)}$$

$$\underline{H}_0 = i \varepsilon_0 \omega \;\; \text{curl}\, \underline{\Pi}_0 \qquad \text{Eq.(7a)}$$

$$\underline{E}_1 = -\gamma_1^2 \underline{\Pi}_1 + \text{grad. div}\, \underline{\Pi}_1 \qquad \text{Eq.(6b)}$$

$$\underline{H}_1 = \sigma \;\text{curl}\, \underline{\Pi}_1 \qquad \text{Eq.(7b)}$$

As a further approximation it is assumed that $\gamma_0 \approx 0$ i.e. that the wavelength in air is infinite. This assumption is justified as long as ranges of interest are small compared to one free space wavelength. In the case on hand where attention is restricted to ranges of a few nautical miles this holds for frequencies up to a few kilocycles per second or, in the transient case, for times longer than about 1 msec. This is the only limit upon the validity of the results which will be derived in this paper. To this approximation Equations (1)-(4) become

$$\Pi_{0x} \approx \frac{I\, d\ell}{2\pi i \varepsilon_0 \omega \gamma^2} \int_0^\infty (u-\lambda)\, e^{\lambda z - uh}\, J_0(\lambda\rho)\, \lambda\, d\lambda \qquad \text{Eq.(8)}$$

$$(z < 0)$$

$$\Pi_{0z} \approx \frac{-I\, d\ell}{2\pi i \varepsilon_0 \omega \gamma^2} \frac{\partial}{\partial x} \int_0^\infty (u-\lambda)\, e^{\lambda z - uh}\, J_0(\lambda\rho)\, d\lambda \qquad \text{Eq.(9)}$$

$$\Pi_{1x} \approx \frac{I\,dl}{4\pi\sigma}\left[\frac{e^{-\gamma R_1}}{R_1} - \frac{e^{-\gamma R_0}}{R_0} + 2\int_0^\infty \frac{e^{-u(z+h)}}{u+\lambda} J_0(\lambda\rho)\,\lambda\,d\lambda\right] \qquad \text{Eq.(10)}$$

$$(z > 0)$$

$$\Pi_{1z} \approx -\frac{I\,dl}{2\pi\sigma\gamma^2}\,\frac{\partial}{\partial x}\int_0^\infty (u-\lambda)\,e^{-u(z+h)} J_0(\lambda\rho)\,d\lambda \qquad \text{Eq.(11)}$$

where the subscripts from γ_1 and u_1 have been dropped for brevity's sake.

Equations (10) and (11) may be written as

$$\Pi_{1x} \approx \frac{I\,dl}{4\pi\sigma}\left[P_1 - P_0 - 2\frac{\partial N}{\partial z} + \frac{2}{\gamma^2}\frac{\partial^2}{\partial z^2}\left(\frac{\partial N}{\partial z} + P_0\right)\right] \qquad \text{Eq.(12)}$$

$$\Pi_{1z} \approx -\frac{I\,dl}{2\pi\sigma\gamma^2}\,\frac{\partial^2}{\partial x\,\partial z}\left(\frac{\partial N}{\partial z} + P_0\right) \qquad \text{Eq.(13)}$$

where the following abbreviations have been used:

$$N = \int_0^\infty \frac{e^{-u(z+h)}}{u} J_0(\lambda\rho)\,d\lambda \qquad \text{Eq.(14)}$$

$$P_0 = \frac{e^{-\gamma R_0}}{R_0} = \int_0^\infty \frac{e^{-u(z+h)}}{u} J_0(\lambda\rho)\,\lambda\,d\lambda \qquad \text{Eq.(15)}$$

$$P_1 = \frac{e^{-\gamma R_1}}{R_1} \qquad \text{Eq.(16)}$$

The second equality in Equation (15) has been demonstrated by Sommerfeld[8) while for N , Magnus and Oberhettinger[9) give the following expression:

$$N = I_0\left[\tfrac{1}{2}\gamma\left\{R_0 - (z+h)\right\}\right]\; K_0\left[\tfrac{1}{2}\gamma\left\{R_0 + (z+h)\right\}\right] \qquad \text{Eq.(17)}$$

8) A. Sommerfeld, op. cit., reference 7, p. 243

9) W. Magnus, F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics. (Chelsea Publishing Co., New York, 1949). p. 133.

where $I_o(z)$ and $K_o(z)$ are modified Bessel functions of order zero. It can furthermore be shown that the functions P_o, P_1 and N all satisfy the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \gamma^2\right) P_o = 0 \qquad \text{Eq.(18)}$$

With the aid of Equations (6), (7), (12) to (16) and (18), the following expressions for the field components may be derived.

$$E_{1x} \approx \frac{I\,d\ell}{4\pi\sigma}\left[-\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(P_1 - P_o) - 2\,\frac{\partial^2 P_o}{\partial z^2} + 2\,\frac{\partial^3 N}{\partial y^2\,\partial z}\right] \qquad \text{Eq.(19)}$$

$$E_{1y} \approx \frac{I\,d\ell}{4\pi\sigma}\left[\frac{\partial^2}{\partial x\,\partial y}(P_1 - P_o) - 2\,\frac{\partial^3 N}{\partial x\,\partial y\,\partial z}\right] \qquad \text{Eq.(20)}$$

$$E_{1z} \approx \frac{I\,d\ell}{4\pi\sigma}\,\frac{\partial^2}{\partial x\,\partial z}(P_1 + P_o) \qquad \text{Eq.(21)}$$

$$H_{1x} \approx -\frac{I\,d\ell}{2\pi\gamma^2}\,\frac{\partial^3}{\partial x\,\partial y\,\partial z}\left(\frac{\partial N}{\partial z} + P_o\right) \qquad \text{Eq.(22)}$$

$$H_{1y} \approx \frac{I\,d\ell}{4\pi}\left[\frac{\partial}{\partial z}(P_1 + P_o) - \frac{2}{\gamma^2}\,\frac{\partial^3}{\partial y^2\,\partial z}\left(\frac{\partial N}{\partial z} + P_o\right)\right] \qquad \text{Eq.(23)}$$

$$H_{1z} \approx \frac{I\,d\ell}{4\pi}\left[\frac{\partial}{\partial y}(P_o - P_1) + 2\,\frac{\partial^2 N}{\partial y\,\partial z} - \frac{2}{\gamma^2}\,\frac{\partial^3}{\partial y\,\partial z^2}\left(\frac{\partial N}{\partial z} + P_o\right)\right] \qquad \text{Eq.(24)}$$

At the interface $z = 0$, all components are continuous except the vertical component of the electric field. Therefore the behaviour of this component at points situated in the non-conducting medium just above the interface cannot be derived from Equation (21). Using Equations (1), (2) and (6a) it may be shown, however, that

$$E_{oz}(z=0) \approx \left[\frac{I\,d\ell}{2\pi\sigma}\cdot\frac{\partial^3 N}{\partial x\,\partial z^2}\right]_{z=0} \qquad \text{Eq.(25)}$$

In deriving Equation (25) one has to wait until the various differentiations have been performed before setting $\gamma_0 \approx 0$, since otherwise incorrect results will be obtained.

All formulas given so far apply to the stationary case, i.e. the case where the antenna current $i(t) = i_o \exp(i\omega t)$. If, however, $i(t)$ is not sinusoidal but a transient, the response of the electromagnetic field to such an antenna current can be found with the aid of Laplace transformation. According to the rules of Laplace transformation the current is considered as a function of frequency $I(i\omega)$. This function is obtained by taking the Laplace transform (symbolically denoted by the operator L) of $i(t)$:

$$I(i\omega) = I(p) = L\{i(t)\} = \int_0^\infty e^{-pt} \, i(t) \, dt$$

where $p = i\omega$, and assuming that $i(t) = 0$ for $t < 0$. After having replaced $i\omega$ by p throughout, Equations (19)-(25) give the field components as a function of p . Their behaviour as a function of time can be obtained by taking the inverse Laplace transform (indicated by the operator L^{-1}) of Equations (19)-(25). If for example $e_{1x}(t)$ denotes the x component of the electric field as a function of time it is obtained from

$$e_{1x}(t) = L^{-1}\{E_{1x}(p)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \, E_{1x}(p) \, dp$$

In this paper the response of the electromagnetic field to a special transient, the step function, will be investigated. In the time domain this function is described by

$$i(t) = I \, . \, u(t)$$

where

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geqslant 0 \end{cases}$$

In the frequency domain it is represented by the function $\frac{I}{p}$. Examination of Equations (19)-(25) reveals that in the expressions for the field components the only quantities occurring are derivatives with respect to the various spatial coordinates of $\frac{P_0}{p}$, $\frac{P_0}{p^2}$, $\frac{P_1}{p}$, $\frac{N}{p}$ and $\frac{N}{p^2}$, since $\gamma^2 = i\sigma\mu_0\omega = \sigma\mu_0 p$

Since the operations of Laplace transformation and differentiation with respect to spatial coordinates commute, it will suffice to compute the inverse Laplace transforms of the aforesaid five quantities. According to Erdélyi et al[10)]

$$L^{-1}\left\{\frac{P_0(p)}{p}\right\} = \frac{1}{R_0}\,\mathrm{Erfc}\,(\lambda_0^{\frac{1}{2}}) \qquad \text{Eq.(26)}$$

$$L^{-1}\left\{\frac{P_1(p)}{p}\right\} = \frac{1}{R_1}\,\mathrm{Erfc}\,(\lambda_1^{\frac{1}{2}}) \qquad \text{Eq.(27)}$$

$$L^{-1}\left\{\frac{P_0(p)}{p^2}\right\} = \frac{t}{R_0}\left[(1+2\lambda_0)\,\mathrm{Erfc}\,(\lambda_0^{\frac{1}{2}}) - \frac{2}{\sqrt{\pi}}\,\lambda_0^{\frac{1}{2}}\,e^{-\lambda_0}\right] \qquad \text{Eq.(28)}$$

where

$$\lambda_0 = \frac{2R_0^2}{T} \qquad \lambda_1 = \frac{2R_1^2}{T} \qquad T = \frac{\vartheta t}{\mu_0\sigma} \qquad \text{Eq.(29)}$$

and

$$\mathrm{Erfc}\ x = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt$$

10) A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Tables of Integral Transforms (McGraw-Hill Book Co. Inc., New York, 1954). Vol. I. pp 245, 246.

In the same table[11)] it is found that

$$L^{-1}\{N(p)\} = \frac{1}{2t} e^{-a\xi} I_0(\xi) \qquad R_0 > 0, \ z+h > 0 \qquad \text{Eq.(30)}$$

where

$$a = 1 + 2\left(\frac{z+h}{\rho}\right)^2 \qquad \text{Eq.(31)}$$

$$\xi = \frac{\mu_0 \sigma \rho^2}{8t} = \frac{\rho^2}{T} \qquad \text{Eq.(32)}$$

Using the well known theorem

$$L^{-1}\{f_1(p) . f_2(p)\} = \int_0^t F_1(t-\tau) F_2(\tau) d\tau$$

where

$$f_1(p) = L\{F_1(t)\}$$

$$f_2(p) = L\{F_2(t)\}$$

one obtains

$$L^{-1}\left\{\frac{N(p)}{p}\right\} = \int_0^t \frac{u(t-\tau)}{2\tau} e^{-a \cdot \frac{\mu_0 \sigma \rho^2}{8\tau}} I_0\left(\frac{\mu_0 \sigma \rho^2}{8\tau}\right) d\tau$$

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11) A. Erdélyi et al., op cit., reference 10, Vol. 1, p. 284.

and

$$L^{-1}\left\{\frac{N(p)}{p^2}\right\} = \int_0^t \frac{(t-\tau)}{2\tau}\, e^{-a\cdot\frac{\mu_0\sigma\rho^2}{8\tau}}\, I_0\left(\frac{\mu_0\sigma\rho^2}{8\tau}\right) d\tau$$

The integrals on the right can be transformed by setting $\frac{\mu_0\sigma\rho^2}{8\tau} = x$ (not to be confused with the coordinate x), thus giving

$$L^{-1}\left\{\frac{N(p)}{p}\right\} = \frac{1}{2} M_{-1}(a,\xi) \qquad \text{Eq.(33)}$$

$$L^{-1}\left\{\frac{N(p)}{p^2}\right\} = \frac{1}{2} t\left[M_{-1}(a,\xi) - \xi M_{-2}(a,\xi)\right] \qquad \text{Eq.(34)}$$

where the function $M_n(a, \xi)$ (n integer) is defined by

$$M_n(a,\xi) = \int_\xi^\infty x^n e^{-ax} I_0(x)\, dx \qquad \text{Eq.(35)}$$

For use in the final results it appears profitable to introduce a related function $N_n(a, \xi)$ defined as

$$N_n(a,\xi) = \int_0^\xi x^n e^{-ax} I_0(x) dx = M_n(a,0) - M_n(a,\xi) \qquad \text{Eq.(36)}$$

$$a > 1$$

The function M (a,0) can again be found in a table of Laplace transforms [12)]

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12) A. Erdélyi et al., op. cit., reference 10, Vol. 1, p. 196.

$$M_n(a,o) = \int_0^\infty x^n e^{-ax} I_o(x)\,dx = \Gamma(n+1)\,(a^2-1)^{-\frac{1}{2}(n+1)} P_n\left(\frac{a}{(a^2-1)^{\frac{1}{2}}}\right) \qquad a>1 \qquad \text{Eq.(37)}$$

In the final results only those functions occur for which n = 0, 1, 2 or 3. For these specific values of n , Equation (37) gives

$$M_o(a,o) = (a^2-1)^{-\frac{1}{2}} = \rho^2(2dR_o)^{-1}$$

$$M_1(a,o) = a(a^2-1)^{-\frac{3}{2}} = \rho^4(\rho^2+2d^2)(2dR_o)^{-3}$$

$$M_2(a,o) = (2a^2+1)(a^2-1)^{-\frac{5}{2}} = \rho^6(3\rho^4+8\rho^2d^2+8d^4)(2dR_o)^{-5}$$

$$M_3(a,o) = 3a(2a^2+3)(a^2-1)^{-\frac{7}{2}} = 3\rho^8(\rho^2+2d^2)(5\rho^4+8\rho^2d^2+8d^4)(2dR_o)^{-7}$$

Eq.(38) $a>1$

where use has also been made of Equation (31) and the quantity z + h has been denoted by d , which notation will be retained throughout the rest of this paper.

Since in Equations (19)-(25) $L^{-1}\{p^{-1}N(p)\}$ and $L^{-1}\{p^{-2}N(p)\}$ do not themselves occur but only their various spatial derivatives, the simplest of these derivatives are given here

$$\frac{\partial M_n}{\partial \rho} = \frac{2}{\rho}\left[2\left(\frac{d}{\rho}\right)^2 M_{n+1} - \xi^{n+1} e^{-a\xi} I_o(\xi)\right]$$

$$\frac{\partial M_n}{\partial z} = -\frac{4d}{\rho^2} M_{n+1}$$

Eq.(39)

in which the arguments have been dropped. From these the higher derivatives may be derived without difficulty. Further properties of the functions $M_n(a, \xi)$ and $N_n(a, \xi)$ are given in Appendices A and B. Since neither of the functions has been tabulated as far as the author has been able to ascertain, the function $N_n(a, \xi)$ has been programmed for computation on the ERA 1101 computer.

1.2 Results for the Field Components

In Section 1.1 the theory has been developed sufficiently far to render possible the computations of the field components, starting from Equations (19) to (25). The actual computations are straightforward but tedious. Therefore only the final results will be given, first for the general case, then for the special case that both dipole and observer are situated at the interface, i.e. z = h = 0 . For the sake of brevity the following abbreviations will be used besides those already defined in Equations (29), (31), (32) and (36):

$$\left.\begin{aligned}
\chi &= \frac{d^2}{T} \\
m(\lambda) &= 3\,\mathrm{Erfc}\,(\lambda^{\frac{1}{2}}) + \frac{2}{\sqrt{\pi}}\lambda^{\frac{1}{2}}(3+2\lambda)\,e^{-\lambda} \\
k(\lambda) &= \frac{4}{\sqrt{\pi}}\lambda^{\frac{3}{2}}\,e^{-\lambda} \\
F(\xi,\chi) &= 2(1+6\chi)N_0 - 4\left(\frac{d}{\rho}\right)^2(8+9\chi)N_1 + 4\left(\frac{d}{\rho}\right)^4(13+4\chi)N_2 - \\
&\qquad - 16\left(\frac{d}{\rho}\right)^6 N_3 + \xi(4\chi-1)e^{-a\xi}I_0(\xi) \\
G(\xi,\chi) &= 2N_0 - 8\left(\frac{d}{\rho}\right)^2 N_1 + 4\left(\frac{d}{\rho}\right)^4 N_2 - 2\xi(1-\chi)\,e^{-a\xi}I_0(\xi) + \\
&\qquad + \xi^2 e^{-a\xi}\left\{I_1(\xi) - I_0(\xi)\right\}
\end{aligned}\right\} \quad \text{Eq. (40)}$$

Using this notation the following expressions are found for the field components (again dropping the arguments of the $N_n(a, \xi)$ in places where no confusion can arise)

$$e_{1x}(t) = \frac{I\,dl}{4\pi\sigma}\left[\frac{2}{R_0^3}\left\{1-3\left(\frac{y}{R_0}\right)^2\right\} + \left\{\left(\frac{y}{R_0}\right)^2-\left(\frac{d}{R_0}\right)^2\right\}\frac{m(\lambda_0)}{R_0^3} + \right.$$

$$+\left\{\left(\frac{x}{R_1}\right)^2-\frac{1}{3}\right\}\frac{m(\lambda_1)}{R_1^3} - \frac{2}{3}\,\frac{k(\lambda_1)}{R_1^3} +$$

$$\left. + \frac{8d}{\rho^4}\left\{-N_0+2\left(\frac{d}{\rho}\right)^2 N_1 + \xi e^{-a\xi} I_0(\xi) + 2\left(\frac{y}{\rho}\right)^2 G(\xi,\chi)\right\}\right] \quad \text{Eq. (41)}$$

$$e_{1y}(t) = \frac{I\,dl}{4\pi\sigma}xy\left[\frac{6-m(\lambda_0)}{R_0^5} + \frac{m(\lambda_1)}{R_1^5} - \frac{16d}{\rho^6}\,G(\xi,\chi)\right] \quad \text{Eq. (42)}$$

$$e_{1z}(t) = \frac{I\,dl}{4\pi\sigma}x\left[\frac{z+h}{R_0^5}\,m(\lambda_0) + \frac{z-h}{R_1^5}\,m(\lambda_1)\right] \quad \text{Eq. (43)}$$

$$h_{1x}(t) = -\frac{I\,dl}{8\pi}xy\left[\frac{5d}{\lambda_0 R_0^5}\left\{3-m(\lambda_0)\right\} + \frac{2d}{R_0^5}\left\{m(\lambda_0)-k(\lambda_0)-\right.\right.$$

$$\left.\left. -6-5\left(\frac{d}{\rho}\right)^2-2\left(\frac{d}{\rho}\right)^4\right\} + \frac{4}{\rho^4\xi}\,F(\xi,\chi)\right] \quad \text{Eq. (44)}$$

$$h_{1y}(t) = \frac{I\,dl}{4\pi}\left[\frac{d}{2\lambda_0 R_0^3}\left\{\left[1-5\left(\frac{y}{R_0}\right)^2\right]\left[3-m(\lambda_0)\right] - 2 - \left(\frac{d}{\rho}\right)^2 - \right.\right.$$

$$\left. -\left(\frac{y}{R_0}\right)^2\left[m(\lambda_0)-k(\lambda_0)-6-5\left(\frac{d}{\rho}\right)^2-2\left(\frac{d}{\rho}\right)^4\right]\right\} - \frac{1}{3}\,\frac{z-h}{R_1^3}\left\{m(\lambda_1)-k(\lambda_1)\right\} +$$

$$\left. + \frac{1}{\xi\rho^2}\left\{(1+6\chi)N_0 - 2\left(\frac{d}{\rho}\right)^2(5+4\chi)N_1 + 8\left(\frac{d}{\rho}\right)^4 N_2 - 2\left(\frac{y}{\rho}\right)^2 F(\xi,\chi)\right\}\right] \quad \text{Eq. (45)}$$

$$h_{1z}(t) = \frac{I\,dl}{4\pi}\, y \left[-\frac{1}{2\lambda_0}\left\{1 - 5\left(\frac{d}{R_0}\right)^2\right\} \frac{m(\lambda_0) - 3}{R_0^3} + \frac{1}{3}\,\frac{m(\lambda_1) - k(\lambda_1)}{R_1^3} - \right.$$
$$- \left(\frac{d}{R_0}\right)^2 \frac{m(\lambda_0) - k(\lambda_0) - 3}{R_0^3} + \qquad \text{Eq.(46)}$$
$$\left. + \frac{4d}{\xi \rho^4}\left\{ \xi N_0 - 2(3 + 5\chi) N_1 + 2\left(\frac{d}{\rho}\right)^2 (9 + 4\chi) N_2 - 8\left(\frac{d}{\rho}\right)^4 N_3 + 2\xi^2 e^{-a\xi} I_0(\xi) \right\} \right]$$

and finally for the vertical electric field in air, measured at points at the surface:

$$e_{0z}(t) = \frac{2\,I\,dl}{\pi \sigma}\,\frac{x}{\rho^4}\left[-\frac{3\rho^4 h}{4 R_0^5} - N_0 + 10\left(\frac{h}{\rho}\right)^2 N_1 - 8\left(\frac{h}{\rho}\right)^4 N_2 + \right.$$
$$\left. + \xi\left\{1 - 4\left(\frac{h}{\rho}\right)^2 \xi\right\} e^{-a\xi}\, I_0(\xi) \right] \qquad \text{Eq.(47)}$$

Several partial checks on the correctness of Equations (41) to (47) are possible. In the first place differentiation of Equations (41) to (43) with respect to time gives the response of the electric field to a delta-impulse excitation. The results thus obtained are essentially identical to Wait's[13)] equations (101) to (103)[*)].

Secondly, Equations (44) to (46) each contain a few terms proportional to $\lambda_0^{-1} = \frac{1}{2} T R_0^{-2}$ which would tend to infinity if $t \longrightarrow \infty$. It can easily be shown, however, that the denominator of these terms is of a degree in T higher than 1 which makes them vanish for $t \longrightarrow \infty$.

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13) J.R. Wait, op.cit., reference 6.

*) A slight difference is found when the derivative of $e_{1y}(t)$ is taken, viz. a factor $xy(z + h)/T^3 t$ is found where Wait gives $xyz/T^3 t$. Since in the expressions Eqs. (10) and (11) for Π_{1x} and Π_{1z} the observer depth z only occurs in the combinations $z + h$ or $z - h$, however, the first factor should be the correct one.

In the third place Equations (41) to (47) each contain a few terms which are independent of T . These terms do not vanish for $t \to \infty$; they give the static fields which would be generated by a horizontal electric dipole fed with direct current. These fields have also been computed by Baños and Wesley[14]; after the necessary modifications in notation and direction of coordinate axes have been made their expressions are identical with those obtained from Equations (41) to (47) by letting $t \to \infty$.

Examination of the behaviour of the solutions for $t \to 0$ (ξ, χ, λ_0 and $\lambda_1 \to 1$) has not much sense because Equations (41) to (47) are not valid for times much shorter than 1 msec. Nevertheless one can easily check, keeping in mind that $N_n(a, \infty) = M_n(a, 0)$ and using Equation (38), that for short times the solutions Equations (41) to (47) tend to zero when $z + h > 0$. When $z = h = 0$ (i.e. $a = 1$) this is not the case anymore, for reasons outlined above.

A special case of interest is formed by the configuration where both transmitting antenna and observer are situated in the interface, i.e. $z = h = 0$, $a = 1$. While deriving the field components the condition $a > 1$ has been encountered several times (cf Equations (30), (31) - (38)). This condition is necessary to ensure the convergence of integrals of the type $\int_0^\infty x^n \exp(-ax) I_0(x) dx$ In the final formulas for the field components, however, the way in which these integrals occur is such that their divergent parts cancel so that their sum remains finite. Since all field components, except E_z , are continuous at the surface and also since no discontinuity in their behaviour is expected when the dipole is raised to the surface (i.e. $h \to +0$), the condition $a > 1$ can be dropped in Equations (41) to (47). The field components then become:

$$e_{1x}(t) = \frac{I\,d\ell}{2\pi\sigma} \frac{1}{\rho^3} \left[1 - 3\left(\frac{y}{\rho}\right)^2 + \frac{1}{3}\left\{ m(2\xi) - k(2\xi) \right\} \right] \qquad \text{Eq.(48)}$$

$$e_{1y}(t) = \frac{3 I\,d\ell}{2\pi\sigma} \frac{xy}{\rho^5} \qquad \text{Eq.(49)}$$

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14) A. Baños jnr., J.P. Wesley, op. cit., reference 5, Chapter IV.

$$e_{1z}(t) = 0 \qquad \text{Eq.(50)}$$

$$h_{1x}(t) = -\frac{I\,dl}{2\pi}\,\frac{xy}{\rho^4}\,e^{-\xi}\left\{I_0(\xi) + 2I_1(\xi)\right\} \qquad \text{Eq.(51)}$$

$$h_{1y}(t) = \frac{I\,dl}{4\pi}\,\frac{1}{\rho^2}\,e^{-\xi}\left[I_0(\xi) + I_1(\xi) - 2\left(\frac{y}{\rho}\right)^2\left\{I_0(\xi) + 2I_1(\xi)\right\}\right] \qquad \text{Eq.(52)}$$

$$h_{1z}(t) = \frac{I\,dl}{16\pi}\,\frac{y}{\rho^3}\left[\frac{3 - m(2\xi)}{\xi} + \frac{4}{3}\left\{m(2\xi) - k(2\xi)\right\}\right] \qquad \text{Eq.(53)}$$

$$e_{oz}(t) = -\frac{2I\,dl}{\pi\sigma}\,\frac{x}{\rho^4}\,\xi\,e^{-\xi}\,I_1(\xi) \qquad \text{Eq.(54)}$$

where use has been made of the relation

$$N_0(1,\xi) = \xi\,e^{-\xi}\left\{I_0(\xi) + I_1(\xi)\right\} \qquad \text{Eq.(55)}$$

a proof of which is given in Appendix B.

1.3. Discussion

The general expressions Equations (41) to (47) are so complicated that it is very difficult to draw any conclusions concerning their behaviour without numerical evaluation of the formulas. Since these computations are very time consuming only two vectors, h_{1x} and e_{oz}, will be examined in more detail here. But first some general remarks can be made.

As was already mentioned the quantities z and h never occur by themselves, but always in the combinations $z + h = d$ or $z - h$. The vector h_{1x} is the only one which contains only $z + h$ and not $z - h$. Therefore in the case of this vector only the sum of the antenna and receiver depth is of importance, not the individual depths themselves.

The angular dependence in the horizontal plane is different for the various vectors. If φ is defined as the angle between the X-axis and the line connecting the projection of P on the XY-plane with the origin, e_{oz} and e_{1z} vary like $\cos\varphi$, h_{1z} like $\sin\varphi$, e_{1y} and h_{1x} like $\sin 2\varphi$ while e_{1x} and h_{1y} display an angular dependence of the form $a + b \cos 2\varphi$.

When looking at Equations (48) to (54), valid in the case that both dipole and observer are situated at the surface, it is seen that all components show a finite rise time except $e_{oz}(t)$ and $e_{1y}(t)$. Of course these vectors will possess a finite, albeit very short, rise time too, since for very short times the assumption $\gamma_0 \approx 0$ is not allowed anymore.

Another quantity of practical interest is the dependence of signal strength upon horizontal range ρ. Here it is already difficult to draw conclusions without numerical computations, as will be seen in the case of the vector e_{oz}. First of all the signal strength is defined as the maximum value which a field component, considered as a function of time, may assume. For most components this will be the value reached as $t \longrightarrow \infty$ but others may reach their maximum earlier and then decrease towards their stationary value as $t \longrightarrow \infty$. Numerical computations show that for $\rho \gg d$ the x component of the magnetic field decreases like the inverse square of range, i.e. $h_{1x}(t) \propto \rho^{-2}$. This is also what one would expect from examination of Equation (44). In Figures 2, 3 and 4, $h_{1x}(t)$ has been plotted for different values of the parameters. (All figures have been plotted on a logarithmic scale because the variables and parameters of interest vary over large ranges. In order to obtain a correct impression of the true shape of the signal, replotting on linear scales is advisable). It is clearly seen from these figures that as ρ increases the field strength needs a longer time to reach its final value: the signal is smeared out. This is also seen from the formulas: time never occurs alone but always in combination with the square of a distance. When all distances are increased by a certain factor time must be

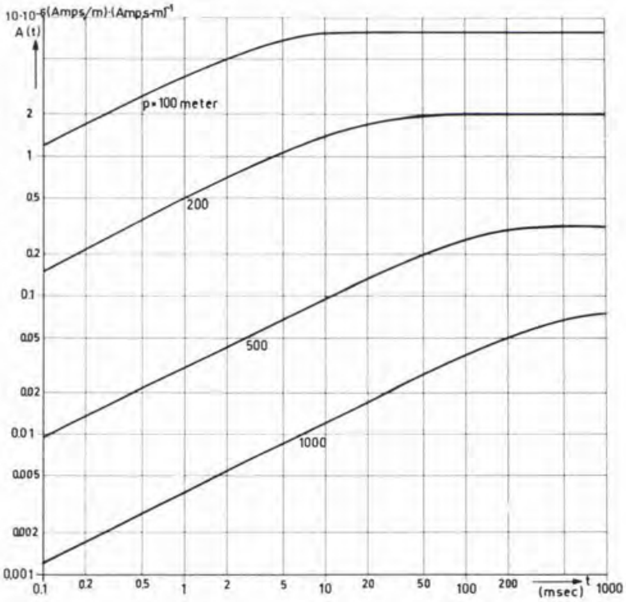

Fig. 2. The x component of the magnetic field $h_{1x}(t) = -I\,d\ell \cdot A(t) \sin 2\varphi$ vs time for various ranges ρ. Antenna and observer at the surface.

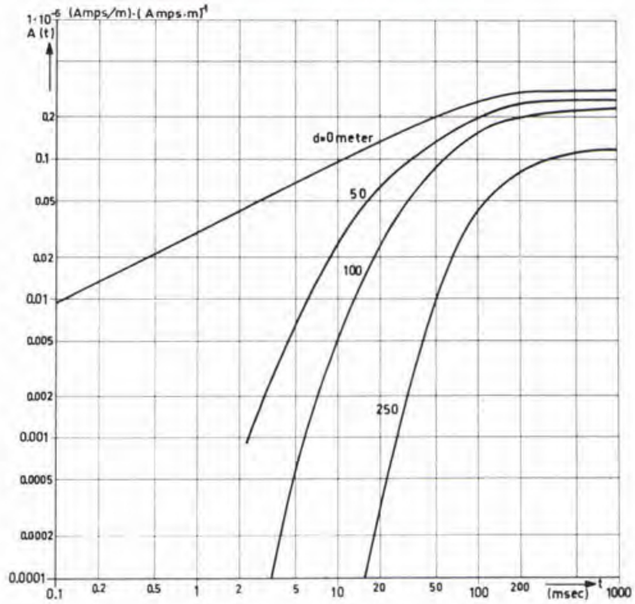

Fig. 3. The component $h_{1x}(t)$ vs time for various values of the combined depth of dipole and observer. Horizontal distance ρ = 500 m.

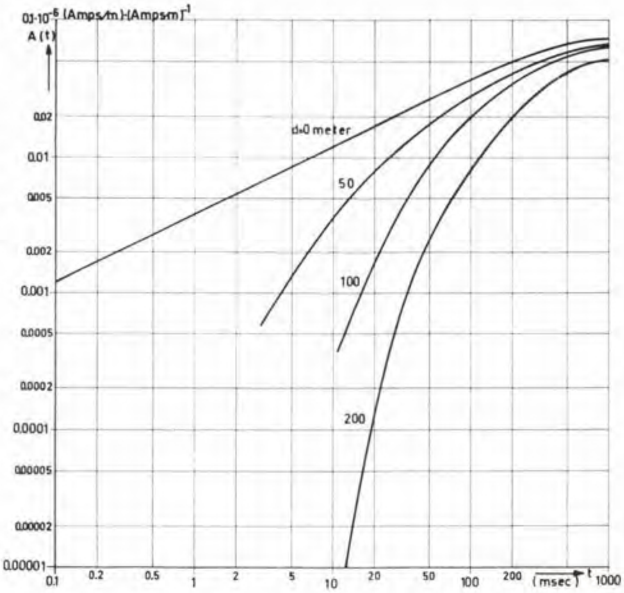

Fig. 4. As Fig. 3., but horizontal distance 1000 m.

increased by the square of that factor to preserve the same signal shape. This smearing out makes detection of the signal more difficult and in the case of communication reduces the rate at which information may be transmitted.

Examination of the formulas without numerical computation may easily lead, however, to incorrect conclusions, as is seen in the case of the vertical electrical vector $e_{oz}(t)$. From Equation (47) one might conclude that $e_{oz}(t) \propto \rho^{-3}$ but numerical computation shows that the maximum value of $e_{oz}(t)$ (i.e. the signal strength) $\propto \rho^{-2}$. Since this vector shows still other interesting properties it is worthwhile to examine its behaviour a little further. In Figures 5 and 6 $e_{oz}(t)$ has been plotted as a function of time for various ranges and antenna depths. It is seen that the maximum value is reached rather quickly after which the field strength decreases again. The time t_{max} at which the maximum is reached can be found by differentiation of Equation (47) with respect to time, from which it results that

$$\xi_{max} = \frac{\rho^2}{4h^2}$$

or $$T_{max} = 4h^2$$

and $$t_{max} = \frac{1}{2}\mu_0 \sigma h^2 .$$

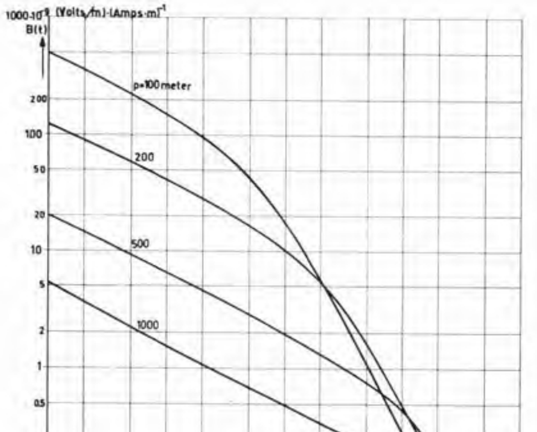

Fig. 5. The vertical electric field in air, $e_{oz}(t) = -I\, d\ell \cdot B(t) \cos\varphi$ vs time for various ranges ρ. Antenna and observer at the surface.

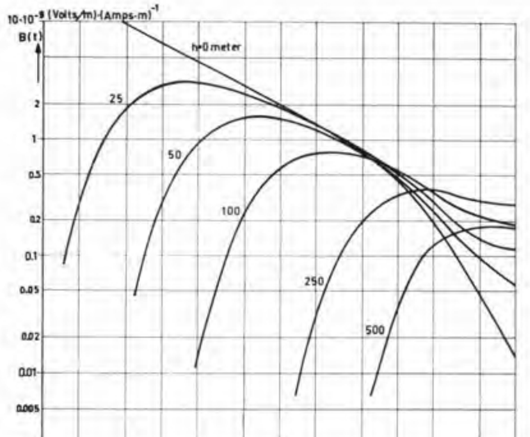

Fig. 6. The electric component $e_{oz}(t)$ vs time for various antenna depths h and observer at the surface. Horizontal distance $\rho = 500$ m.

Thus t_{max} only depends on the depth of the antenna, not on horizontal range, and the pulse is not smeared out as is the case with the other components. The way in which the height of the maximum of $e_{oz}(t)$ depends on ρ cannot be determined analytically but must be computed by numerical means. The maximum value of the part of Equation (47) within the square brackets is denoted by $Q(^h/\rho)$ and has been plotted vs $^h/\rho$ in Figure 7. It is seen that for $\frac{h}{\rho} \lesssim 0,3$, i.e. $\rho \gtrsim 3h$, the quantity $Q(^h/\rho)$ is proportional to $(^h/\rho)^{-1}$, so that the maximum value of $e_{oz}(t)$ itself decreases like the inverse square of range. For points of observation closer to the dipole a different range dependence of the signal strength (a more rapid fall-off) is found. Figure 8 shows a plot of $e_{oz}(t)$ for a fixed antenna depth and various ranges which illustrates the fact that t_{max} does not depend on horizontal range.

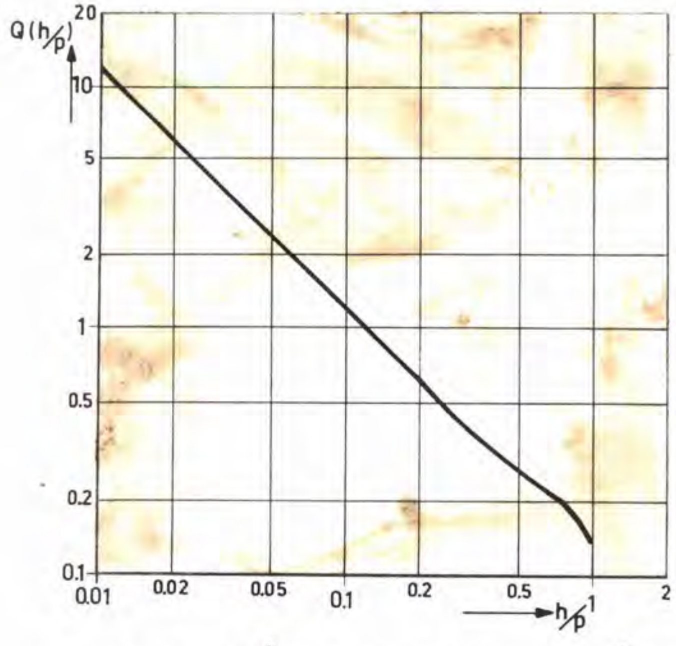

Fig. 7. $Q(^h/\rho)$ as a function of $^h/\rho$. See text.

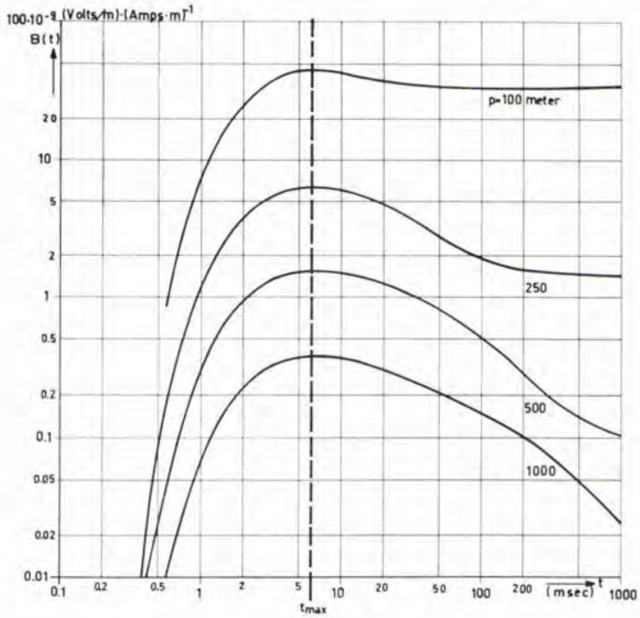

Fig. 8. The electric component $e_{oz}(t)$ vs time for a fixed antenna depth of 50 m. and various ranges ρ. Observer at the surface. The abscissa of the maxima is independent of ρ.

Table I summarizes the results discussed above and for the sake of comparison also lists the range dependences for sinusoidal signals[15)], the frequency of which satisfies the inequalities $\gamma_0 \rho \ll 1$ and $\gamma_1 \rho \gg 1$. (For a horizontal range of a few thousand meters this means any frequency between about 10 and 5000 cps).

Component	Step-function	Sine-wave
e_{oz}	ρ^{-2}	ρ^{-2}
h_{1x}	ρ^{-2}	ρ^{-3}

Table I

Range dependence of signal strength for a step-function vs. sine-wave signal.

From this table it is seen that the vertical electric vector in air, e_{oz}, possesses very interesting properties with regard to the transmission of information from a submerged dipole to a receiver at the surface of the sea. For sine-wave signals the attenuation of e_{oz} with horizontal range is less than for any of the other electric or magnetic components[15)]. In the case of step-function-like signals the attenuations of e_{oz} and h_{1x} with horizontal distance are equal but the electric signal $e_{oz}(t)$ is not smeared out, as is the case with the magnetic signal $h_{1x}(t)$.

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15) A. Baños jr., J.P. Wesley, op. cit., reference 5, Vol. I, p. 199 and Vol. II, p. 126.

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SECTION 2. RECEPTION OF TRANSIENT SIGNALS

2.1. Possible Detectors.

In Section 1 it was seen that during the propagation over a conducting half-space not only the amplitude but also the shape of a transient is changed. In this section the influence of the equipment used for reception of the transient will be investigated.

In the receiving equipment three parts can be distinguished:

(a) detector

(b) amplifiers and signal processing equipment

(c) display equipment.

Considered from the point of signal shape deformation part (c) will not present many difficulties. It will usually consist of a recorder with a flat characteristic in the frequency range of interest. Part (b) may consist of anything from a simple high-gain amplifier to a complicated signal processing equipment and therefore will not be treated further. Part (a) can be divided into detectors for the reception of electric signals (antennas) and detectors which detect variations in the magnetic field. In this section attention will be focussed on detectors of the second kind, since they are widely used and their response may be strongly frequency dependent.

The frequency dependence of magnetic detectors depends on the type of detector used. The response of alkaline vapour magnetometers, for instance, is usually flat within the frequency range of interest. Such a magnetometer will not alter the shape of a transient received by it and therefore need not be considered further here. Frequently, however, coils are used for the detection of changes in the magnetic field. Their frequency response will depend on the material used for the core and on the coil impedance as compared with the input impedance of the amplifier following the coil. In some cases the input impedance of the amplifier is very high so that the coil impedance has no influence. For a harmonic field variation an air-cored coil will then give an e.m.f. proportional to frequency according to the law of induction

$$e \propto -\frac{dB}{dt} = i\omega B$$

If the field variation is a transient the output signal from the coil will be proportional to the time derivative of the transient. A magnetic field varying like a step-function will thus give rise (in the idealized case) to a delta-impulse at the output of the coil. The mathematical description of this case presents no difficulties.

Things become more complicated when a core of ferromagnetic material is used, especially when this core is not laminated and eddy currents can occur. This is, for instance, the case with the mine coils used by the Electromagnetic Group at this Center in propagation experiments. Experimentally it has been found that the frequency response of these coils in the frequency range of interest is roughly proportional to $\omega^{\frac{1}{2}}$. The problem of determining the response of these coils to transient variations in the magnetic field may be solved analytically but there are some mathematical difficulties. This case will therefore be examined here in somewhat more detail.

2.2. Transient response of coils with a frequency characteristic proportional to $\omega^{\frac{1}{2}}$.

The computation of the transient response of a coil with a frequency characteristic proportional to $\omega^{\frac{1}{2}}$ to simple field changes does not in principle present great difficulties. Denoting the frequency spectrum of the input signal (i.e. the change in the magnetic field) by Φ (p), the spectrum F(p) of the output signal (i.e. the emf generated by the coil) is found by multiplying the input spectrum by $Sp^{\frac{1}{2}}$ (where S denotes the sensitivity of the coil)

$$F(p) = S\, p^{\frac{1}{2}}\, \Phi(p)$$

The output signal in the time domain f(t) is then found by taking the inverse Laplace transform of F(p).

$$f(t) = L^{-1}\left\{F(p)\right\} = S\, L^{-1}\left\{p^{\frac{1}{2}}\, \Phi(p)\right\}$$

e.g. the response of the coil to a unit step at t = 0 in the magnetic field is found from

$$f(t) = S.L^{-1}\left\{p^{\frac{1}{2}}.p^{-1}\right\} = S.L^{-1}\left\{p^{-\frac{1}{2}}\right\} = S.\pi^{-\frac{1}{2}}\, t^{-\frac{1}{2}} \qquad \text{Eq.(56)}$$

which is a pulse with an infinitely steep leading edge and a comparatively slow fall off proportional to $t^{-\frac{1}{2}}$ (Figure 9).

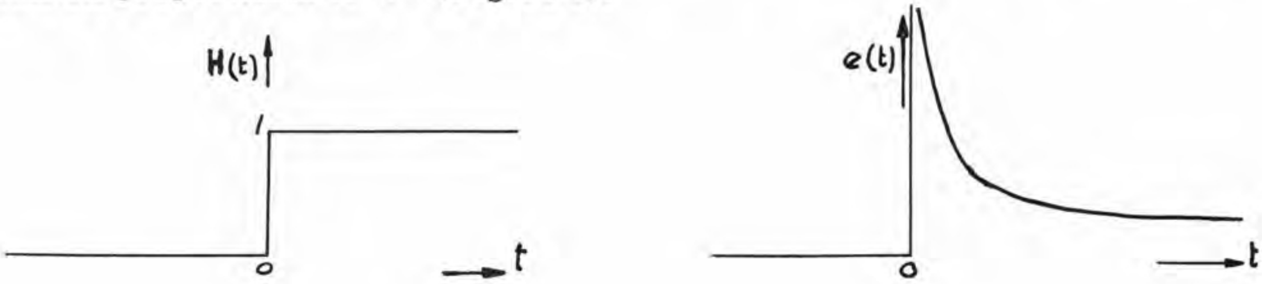

Fig. 9. Response of a coil with frequency characteristic $\propto \omega^{\frac{1}{2}}$ to a unit step in the magnetic field.

It is of practical interest to compute the output signal for the case in which the coil is being used to detect a signal from a horizontal electric dipole excited by a step-function current. To this end it is assumed that both coil and dipole are situated at the surface of the sea at a distance ρ from each other, and that the line connecting them makes an angle φ with the dipole which is supposed to be oriented along the X-axis. Moreover it is assumed that the coil is oriented in a direction parallel to the X-axis, so that the change in the magnetic field detected by the coil is given by Equation (51) for $h_{1x}(t)$. In polar coordinates this equation is

$$h_{1x}(t) = -\frac{I\,dl}{4\pi}\,\frac{\sin 2\varphi}{\rho^2}\,e^{-\xi}\left\{I_0(\xi) + 2I_1(\xi)\right\} \qquad \text{Eq. (51a)}$$

where as before

$$\xi = \frac{\mu_0 \sigma \rho^2}{8t}$$

In principle the response e(t) of the coil to this signal may be computed from

$$e(t) = S . L^{-1}\left[p^{\frac{1}{2}} L\left\{h_{1x}(t)\right\}\right]$$

The computation of the inverse Laplace transform of the expression within the square brackets is, however, very complicated. Therefore a different method will be followed here. This method has the additional advantage of providing, at an intermediate stage, an integral representation for e(t) which for large ξ (short times) can be expanded asymptotically, thus providing information concerning the behaviour of e(t) for small t. The final result will be an expression of e(t) in terms of generalized hypergeometric functions.

To this end use is made of the theorem mentioned on page 10. Denoting the inverse Laplace transform of the function $p^{\frac{1}{2}}$ by $f_1(t)$, i.e.

$$f_1(t) = L^{-1}\left\{p^{\frac{1}{2}}\right\}$$

the theorem would run in this case

$$e(t) = S . L^{-1}\left[p^{\frac{1}{2}} L\left\{h_{1x}(t)\right\}\right] = S\int_0^t f_1(t-\tau)\, h_{1x}(\tau)\, d\tau \qquad \text{Eq.(57)}$$

Unfortunately it turns out that from the straightforward application of this method a divergent integral results. The reason for this is that the inverse transform $L^{-1}\left\{p^{\frac{1}{2}}\right\}$ does not exist in itself but only as the limiting case $\beta \to 0$ of the more general transform[16)]

$$L^{-1}\left\{p^{\frac{1}{2}} e^{-\beta|p|}\right\} = 2^{-\frac{3}{2}} \pi^{-\frac{1}{2}} (\beta^2+t^2)^{-\frac{3}{2}} \left[(\beta^2+t^2)^{\frac{1}{2}} - 2t\right]\left[(\beta^2+t^2)^{\frac{1}{2}} + t\right]^{\frac{1}{2}} \qquad \text{Eq.(58)}$$

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16) G.A. Campbell, R.M. Foster, Fourier Integrals for Practical Applications (D. van Nostrand Co.Inc., Princeton, N.J. 1957). 4th ed. p. 75.

The correct procedure would be to insert Equation (58) into the integral in Equation (57) and integrate before letting $\beta \rightarrow 0$, but then the integral becomes too complicated to solve.

The problem can be solved, however, in the following way. The coil (Figure 10a) is replaced by two systems, connected in series (Figure 10b).

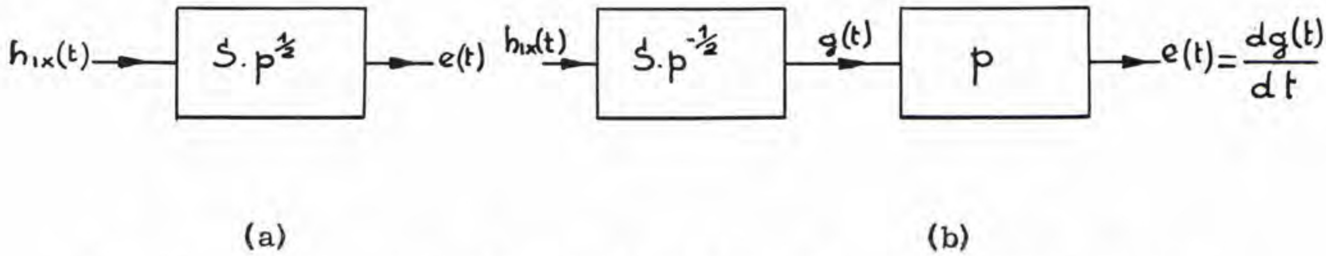

Fig. 10. Replacement of the coil by a series connection of two other systems which together have the same frequency characteristic.

The frequency characteristic of the first system is given by the function $S \cdot p^{-\frac{1}{2}}$, while the second system is a normal differentiator, having a frequency characteristic p. The net effect of the two systems in series is therefore the same as that of the coil with frequency characteristic $S \cdot p^{\frac{1}{2}}$. The inverse transform of the function $p^{-\frac{1}{2}}$ exists and has already been given in Equation(56). The signal at the output of the first system is then equal to

$$g(t) = S \pi^{-\frac{1}{2}} \int_0^t (t-\tau)^{-\frac{1}{2}} h_{1x}(\tau) \, d\tau \qquad \text{Eq.(59)}$$

This integral converges. The output signal e(t) of the second system, which is identical to the signal obtained at the output of the coil, is then found by differentiating g(t) once with respect to time, an operation which can be performed entirely in the time domain.

After this outline of the procedure to be followed the signal e(t) is computed as follows. Substitution of Equation (51a) into Equation (59) gives for g(t)

$$g(t) = -S\,\frac{I\,dl}{4\pi}\,\frac{\sin 2\varphi}{\rho^2}\,\pi^{-\frac{1}{2}}\int_0^t (t-\tau)^{-\frac{1}{2}}\,e^{-\frac{\mu_0\sigma\rho^2}{8\tau}}\left\{I_0\left(\frac{\mu_0\sigma\rho^2}{8\tau}\right)+2I_1\left(\frac{\mu_0\sigma\rho^2}{8\tau}\right)\right\}d\tau$$

$$= C(\rho,\varphi)\,\xi^{\frac{1}{2}}\int_\xi^\infty x^{-\frac{3}{2}}(x-\xi)^{-\frac{1}{2}}\,e^{-x}\left\{I_0(x)+2I_1(x)\right\}dx \qquad \text{Eq.(60)}$$

where

$$x = \frac{\mu_0\sigma\rho^2}{8\tau} \qquad\qquad C(\rho,\varphi) = -\frac{I\,dl}{4\pi}\,\frac{\sin 2\varphi}{\rho^2}\left(\frac{\mu_0\sigma\rho^2}{8\pi}\right)^{\frac{1}{2}}\cdot S$$

Before solving this integral exactly it is worthwhile to investigate its behaviour for large ξ , since, from this, information can be obtained concerning the behaviour of e(t) for small t.

For large values of their argument the modified Bessel functions in Equation(60) may be replaced by the first few terms of their asymptotic expansions

$$I_0(x) \sim \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left(1+\frac{1}{8x}+\frac{9}{128x^2}+\cdots\right)$$

$$I_1(x) \sim \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left(1-\frac{3}{8x}-\frac{15}{128x^2}-\cdots\right) \qquad \text{Eq.(61)}$$

Inserting this into Equation (60) an asymptotic expansion for g(t) is obtained

$$g(t) \sim (2\pi)^{-\frac{1}{2}}C(\rho,\varphi)\,\xi^{\frac{1}{2}}\int_\xi^\infty x^{-2}(x-\xi)^{-\frac{1}{2}}\left(3-\frac{5}{8x}-\frac{21}{128x^2}-\cdots\right)dx$$

$$\sim \frac{3}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}C(\rho,\varphi)\,\xi^{-1}\left(1-\frac{5}{32\xi}-\frac{35}{1024\xi^2}-\cdots\cdots\right) \qquad \text{Eq.(62)}$$

and after differentiation with respect to time

$$e(t) = \frac{dg(t)}{dt} \sim \frac{3}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}C(\rho,\varphi)\frac{8}{\mu_0\sigma\rho^2}\left(1-\frac{5}{16\xi}-\frac{105}{1024\xi^2}-\cdots\right)$$

$$\sim -\frac{I\,dl}{4\pi}\,\frac{\sin 2\varphi}{\rho^3}\,\frac{3}{(\mu_0\sigma)^{\frac{1}{2}}}\,S\left(1-\frac{5}{16\xi}-\frac{105}{1024\xi^2}-\cdots\right) \qquad \text{Eq.(63)}$$

With the aid of Equation (32) this expression can be written as a series in ascending powers of t :

$$e(t) \sim -\frac{I\,d\ell}{4\pi}\,\frac{\sin 2\varphi}{\rho^3}\,\frac{3}{(\mu_0\sigma)^{\frac{1}{2}}}\,S\left\{1-\frac{5}{2}\,\frac{t}{\mu_0\sigma\rho^2}-\frac{105}{16}\left(\frac{t}{\mu_0\sigma\rho^2}\right)^2-\cdots\right\} \qquad \text{Eq. (64)}$$

For large times it is profitable to express e(t) in terms of Meyer's G-functions, which can be expanded into a series of ascending powers of ξ . (The most important properties of these G-functions, which are a generalization of the hypergeometric function, are listed in Appendix C). To that end one uses the relation[17)]

$$\int_\xi^\infty x^{-\frac{3}{2}}\,e^{-x}\,I_\nu(x)(x-\xi)^{-\frac{1}{2}}\,dx = 2^{\frac{3}{2}}\xi^{\frac{1}{2}}\,G_{23}^{21}\left(2\xi\,\middle|\,\begin{matrix}-1 & 0 & \\ -\frac{1}{2} & \nu-\frac{3}{2} & -\nu-\frac{3}{2}\end{matrix}\right) \qquad \text{Eq. (65)}$$

which upon substitution in Equation (60) gives

$$g(t) = 2^{\frac{3}{2}}\,C(\rho,\varphi)\,\xi\left[G_{23}^{21}\left(2\xi\,\middle|\,\begin{matrix}-1 & 0 & \\ -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2}\end{matrix}\right)+2\,G_{23}^{21}\left(2\xi\,\middle|\,\begin{matrix}-1 & 0 & \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{2}\end{matrix}\right)\right] \qquad \text{Eq. (66)}$$

Differentiating this expression once with respect to time one obtains for e(t)

$$\begin{aligned} e(t) &= \frac{4\sqrt{2}}{\mu_0\sigma\rho^2}\,C(\rho,\varphi)\left[G_{23}^{21}\left(2\xi\,\middle|\,\begin{matrix}1 & 1 & \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\end{matrix}\right)+2\,G_{23}^{21}\left(2\xi\,\middle|\,\begin{matrix}1 & 1 & \\ \frac{3}{2} & \frac{3}{2} & -\frac{1}{2}\end{matrix}\right)\right] \\ &= -\frac{I\,d\ell}{4\pi}\,\frac{\sin 2\varphi}{\rho^3}\,\frac{2S}{(\mu_0\sigma\pi)^{\frac{1}{2}}}\left[G_{23}^{21}\left(2\xi\,\middle|\,\begin{matrix}1 & 1 & \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\end{matrix}\right)+2\,G_{23}^{21}\left(2\xi\,\middle|\,\begin{matrix}1 & 1 & \\ \frac{3}{2} & \frac{3}{2} & -\frac{1}{2}\end{matrix}\right)\right] \end{aligned} \qquad \text{Eq. (67)}$$

For a numerical computation of e(t) it is necessary to expand the G-function into a power series, giving

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17) A. Erdélyi et al., op. cit., reference 10, Vol. II, p. 207

$$e(t) = -\frac{I\,dl}{4\pi}\,\frac{\sin 2\varphi}{\rho^3}\,\frac{2S}{(\mu_0\sigma)^{\frac{1}{2}}}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\xi^{\frac{1}{2}}\left[1+\frac{1}{\pi}\sum_{n=1}^{\infty}(-)^{n-1}\frac{\Gamma^2(n+\frac{1}{2})}{\Gamma^2(n+1)\Gamma(n+2)}\,(2\xi)^n\cdot\right.$$
$$\left.\cdot\left\{\frac{n^2+2n-1}{n+1}+n(n-1)\Big[\ln(2\xi)+2\psi(n+\tfrac{1}{2})-3\psi(n+1)\Big]\right\}\right] \qquad \text{Eq.(68)}$$

where the function $\psi(z)$ is the logarithmic derivative of the gamma function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

A more detailed derivation of these results is given in Appendix D.

From Equation (68) it is seen that $e(t) = O(\xi^{\frac{1}{2}})$ as $\xi \longrightarrow 0$, i.e. that $e(t)$ approaches zero like $t^{-\frac{1}{2}}$ as $t \longrightarrow \infty$. With the aid of Equations (64) and (68) it is now possible to draw a qualitative sketch of the signal as it may be measured at the output of the mine coil (see Figure 11. In reality the pulse will of course have a finite rise-time because at frequencies above a few hundred cycles per second the response of the coil is no longer proportional to $\omega^{\frac{1}{2}}$, but flattens out.)

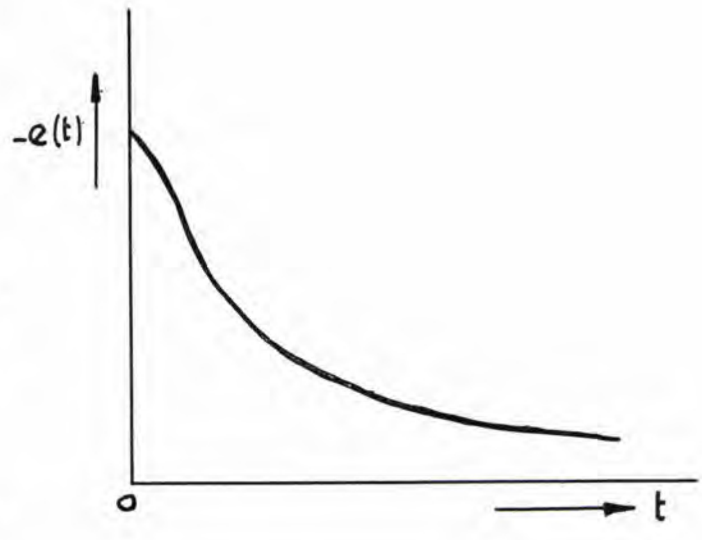

Fig. 11. Shape of the signal $-e(t)$ at the output of a detector coil with frequency characteristic $\propto \omega^{\frac{1}{2}}$.

Another important feature of $e(t)$ which follows from Equations (64) and (68) is its dependence upon horizontal range ρ. The field variation $h_{1x}(t)$ giving rise to the emf $e(t)$ varies itself as the inverse square of distance but the emf $e(t)$ generated by it varies as the inverse cube of ρ. At first sight this seems paradoxical. The explanation is that during propagation of the transient along the surface of the conducting medium the higher frequencies at first present in the signal are attenuated more quickly than the lower ones. During detection of this signal with a coil which has a frequency characteristic proportional to a positive power of the frequency, the lower frequencies are less well transferred by the coil than are the higher frequencies. These combined effects result in a higher attenuation of the transient as a whole than would be expected on the basis of the propagation laws alone. In the case of a coil giving an emf $e(t) \propto \dot{H}$ this extra attenuation is even more serious. Since $d/dt = -8(\mu_0 \sigma \rho^2)^{-1} \xi^2 \, d/d\xi$ the influence of such a coil on the range dependences of the various field components, as for instance given in Table I for the case of h_{1x}, can be described as multiplication of these range dependences by a factor ρ^{-2}. The same holds of course for differentiating electronic equipment placed behind the detection coil. On the other hand, it is seen from Equation (66) that a characteristic $\propto \omega^{-\frac{1}{2}}$ would give an overall range dependence $\propto \rho^{-1}$ for $h_{1x}(t)$, i.e. a reduction of the attenuation.

The truth of these statements is in fact only apparent in the case $d = 0$, i.e. the case where the signal depends only on ρ and t. If $d \neq 0$ matters become more complicated since the signal received does not depend on d itself but rather on the ratio d/ρ, which ratio decreases when ρ is increased while d is kept fixed. This effect may partly compensate the increased attenuation with range caused by the detection coil. This behaviour is more or less analogous to that of the vertical electric field $e_{oz}(t)$ where as one has seen, the decrease of the ratio d/ρ reduces the attenuation with range from an inverse cube to an inverse square law. For a more quantitative description of these phenomena a detailed numerical analysis of the problem is necessary, which will be refrained from here.

The result derived above concerning the change in range dependence of transients received by detectors having frequency characteristics $\propto \omega^{\frac{1}{2}}$, $\propto \omega$ or $\propto \omega^{-\frac{1}{2}}$ suggest the following more general hypothesis:

If a transient disturbance of the electromagnetic field which is propagated over a highly conducting half-space, and which depends on no other distance than the horizontal distance ρ between antenna and dector and of which the amplitude is proportional to $\rho^{-\alpha}$, is detected with the aid of a detector having a frequency characteristic proportional to ω^{β}, the output of the detector will depend on horizontal range as $\rho^{-(\alpha+2\beta)}$.

The general validity of this statement, however, has not yet been proven.

SUMMARY

For the case of a horizontal electric dipole situated in a conducting half-space and excited by a step-function current, expressions are derived for the various components of the electromagnetic field as a function of time for points of observation both in the conducting half-space and at the interface. It is shown that the attenuation with horizontal range of transient signals may differ from the attenuation experienced by pure sinusoidal signals. While all components exhibit some degree of "smearing out" of the signal as it propagates, the vertical electric component in air exhibits some advantages in this regard over the other components. For the vertical component the time for the signal to reach maximum value is shown to be independent of the horizontal distance between source and receiver which makes this component very suitable for transfer of information. In this analysis propagation time has been neglected.

Furthermore, an investigation is made of the influence of the frequency characteristic of magnetic detectors on the shape of the received signal. It is shown that a rising frequency characteristic increases the overall attenuation of the final signal with horizontal range while a falling characteristic reduces this attenuation.

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APPENDIX A. Approximate Formulas for $M_n(a, \xi)$ and $N_n(a, \xi)$.

Since tables of the functions $M_n(a, \xi)$ and $N_n(a, \xi)$ do not exist, some work has been done in order to find approximate expressions for these functions in terms of tabulated functions. The results of this work are given in this Appendix and in Appendix B. Since $M_n(a, \xi)$ and $N_n(a, \xi)$ are related through Equation (36), a result which is derived for $M_n(a, \xi)$ can also be used for the numerical computation of $N_n(a, \xi)$ and vice versa.

A.1. Approximation of $M_n(a, \xi)$ for large ξ.

When ξ is sufficiently large the modified Bessel function $I_o(x)$ in the integrand of

$$M_n(a, \xi) = \int_\xi^\infty x^n e^{-ax} I_o(x)\, dx \qquad \text{Eq.(A.1)}$$

can be replaced by the first term of its asymptotic expansion:

$$I_o(x) \sim (2\pi x)^{-\frac{1}{2}} e^x$$

Substituting this in Equation (A.1) and using the definition of the incomplete gamma function $\Gamma(b, z)$[18] one obtains

$$M_n(a, \xi) \sim (2\pi)^{-\frac{1}{2}} (a-1)^{-n-\frac{1}{2}} \Gamma\left\{n+\tfrac{1}{2}, (a-1)\xi\right\} \qquad \text{Eq.(A.2)}$$

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18) A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi Higher Transcendental Functions (McGraw-Hill Book Company Inc., New York, 1953). Vol. II, pp. 133 ff.

This relation can be simplified by the use of the recurrence relation

$$\Gamma(b+1, z) = b\,\Gamma(b, z) + z^b e^{-z} \qquad \text{Eq.(A. 3)}$$

and the formula* [19]

$$\Gamma(\tfrac{1}{2}, z^2) = \pi^{\frac{1}{2}}\ \mathrm{Erfc}\ z \qquad \text{Eq.(A. 4)}$$

Repeated substitution of Equations (A. 3) and (A. 4) into Equation (A. 2) gives finally with the aid of the first of Equations (40) :

$$M_n(a,\xi) \sim 2^{-n-1}\pi^{-\frac{1}{2}}\left(\frac{d}{\rho}\right)^{-2n-1}\left[\Gamma(n+\tfrac{1}{2})\,\mathrm{Erfc}(2\lambda)^{\frac{1}{2}} + \frac{(2\lambda)^n - 1}{2\lambda - 1}\,(2\lambda)^{\frac{1}{2}} e^{-2\lambda}\right] \qquad \text{Eq.(A. 5)}$$

This expression can be used for the numerical computation of $M_n(a, \xi)$ for large values of ξ . Table II gives an idea of the accuracy so obtained. It is seen that the relative error is smallest for high n and a close to unity.

	a = 1.02			a = 1.2		
	Computer	Eq.(A. 5)	Relative difference	Computer	Eq.(A. 5)	Relative difference
$M_0(a,10)$	2.6486	2.6355	0.50%	0.07266	0.07195	0.98%
$M_1(a,10)$	117.89	117.53	0.31%	1.0430	1.0335	0.91%

Table II. Comparison between the exact values of $M_n(a, \xi)$ and those computed using Eq.(A. 5) for various values of the parameters.

* The definition of the error function as used in this paper differs in a factor $2\pi^{-\frac{1}{2}}$ from the one used by Erdélyi et al in reference 18, but is the same as the one used by Erdélyi et al in 10.

19) A. Erdélyi et al., op. cit., reference 18, Vol. II, p. 147.

A. 2. Series expansion of $N_n(a, \xi)$.

When ξ is finite $I_o(x)$ can be replaced in the integrand of $\int_0^\xi x^n e^{-ax} I_o(x)dx$ by its series expansion and this form integrated term by term. The result is a series of incomplete gamma functions,

$$\int_0^\xi x^n e^{-ax} I_o(x)\,dx = \sum_{k=0}^{\infty} \int_0^\xi x^n e^{-ax} \frac{(\frac{1}{2}x)^{2k}}{(k!)^2}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{2k}(k!)^2} \int_0^\xi x^{n+2k} e^{-ax}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{2k}(k!)^2}\, a^{-n-2k-1}\, \gamma(n+2k+1, a\xi) \qquad \text{Eq. (A. 6)}$$

where the incomplete gamma function $\gamma(b + 1,z)$ is defined by

$$\gamma(b+1, z) = \int_0^z e^{-t}\, t^b\, dt$$

In case b is an integer $\gamma(b + 1,z)$ can be expressed as[20)]

$$\gamma(b+1, z) = b!\left[1 - e^{-z}.\, e_b(z)\right]$$

where $e_b(z)$ is the truncated exponential series

$$e_b(z) = \sum_{m=0}^{b} \frac{z^m}{m!}$$

Inserting this into Equation (A. 6) gives:

$$N_n(a, \xi) = \sum_{k=0}^{\infty} \frac{(n+2k)!}{2^{2k}(k!)^2}\; \frac{1 - e^{-a\xi}.\, e_{n+2k}(a\xi)}{a^{n+2k+1}} \qquad \text{Eq. (A. 7)}$$

E.G. for n = 0 the first few terms are

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20) A. Erdélyi et al., op. cit., reference 18, Vol. II, p. 136

$$N_o(a,\xi) = \frac{1-e^{-a\xi}}{a} + \frac{1}{2}\cdot\frac{1-\left(1+a\xi+\frac{1}{2}a^2\xi^2\right)e^{-a\xi}}{a^3} +$$

$$+\frac{3}{8}\,\frac{1-\left(1+a\xi+\frac{1}{2}a^2\xi^2+\frac{1}{6}a^3\xi^3+\frac{1}{24}a^4\xi^4\right)e^{-a\xi}}{a^5} + \ldots \qquad \text{Eq.(A.8)}$$

This series converges extremely rapidly for small ξ, while its rate of convergence for ξ of the order of unity is still reasonably high. Taking for example $a = 1$, $\xi = 0.5$ and using the first two terms of Equation (A.8) gives $N_o(1, 0.5)$ with an accuracy of 2.5 parts in 10^4. Three terms give a six decimal accuracy.

Expansion of $N_n(a, \xi)$ in a double series of ascending powers of ξ and $a - 1$ is also possible but the rate of convergence is greatly inferior to that of Equation (A.7). Therefore the derivation of this series will not be given here.

In order to facilitate the numerical computation of the field components a program has been written for computation of the function $N_n(a, \xi)$ on the ERA 1101 computer. Tables have been made which give $N_n(a, \xi)$ to four significant figures for the following values of the parameters and variables:

n = 0, 1, 2 and 3

ξ = 0(0.02) 1 and 0(0.1) 10

a = 1(0.001) 1.01 (0.01) 1.2 (0.1) 3.

Figures 12 and 13 give an impression of the behaviour of the functions $N_o(a, \xi)$ and $N_1(a, \xi)$.

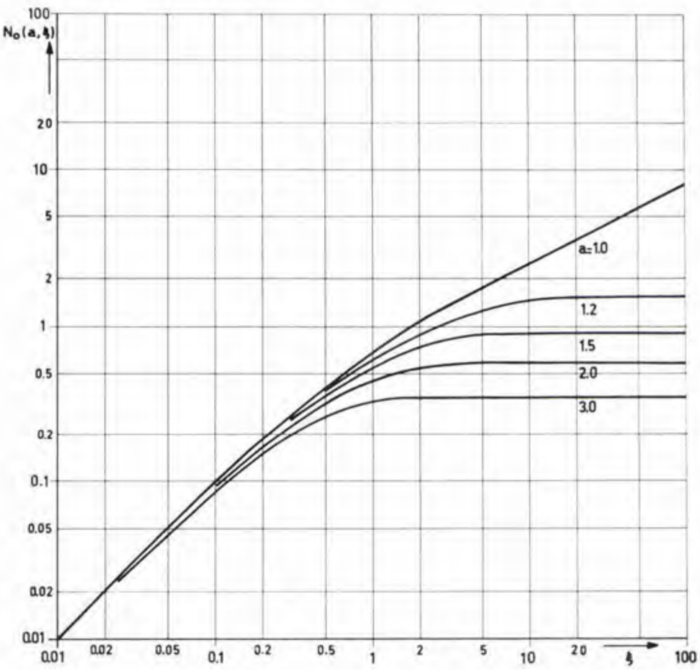

Fig. 12. The function $N_o(a, \xi)$ vs ξ for various values of a.

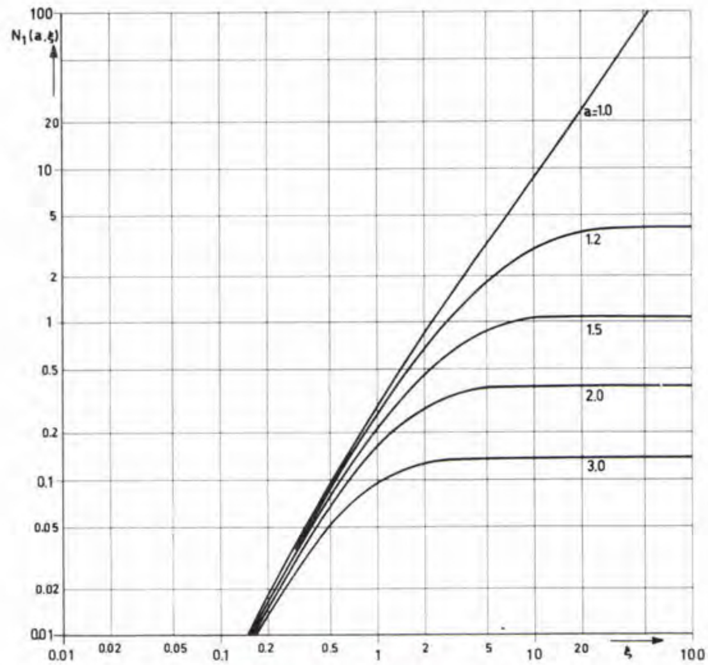

Fig. 13. The function $N_1(a, \xi)$ vs ξ for various values of a.

APPENDIX B. Proof of the relation $N_o(1, \xi) = \xi e^{-\xi}\left\{I_o(\xi) + I_1(\xi)\right\}$

In this Appendix the following relation will be shown to hold:

$$N_o(1,\xi) = \xi e^{-\xi}\left\{I_o(\xi) + I_1(\xi)\right\} \qquad \text{Eq.(B.1)}$$

which makes it possible to express the field components in the case h=z=0 in a closed form.

As a first step in the proof it may be shown that

$$N_o(1,\xi) = \xi \,.\, {}_1F_1\left(\tfrac{1}{2}; 2; -2\xi\right) \qquad \text{Eq.(B.2)}$$

where ${}_1F_1$ (a; b; x) denotes the confluent hypergeometric function. For this it is sufficient to consider $N_o(1, \xi)$ as a special case of the following Riemann-Liouville integral[21)]

$$[\Gamma(\mu)]^{-1}\int_0^{\xi} x^{\nu} e^{-ax} I_{\nu}(ax)(\xi - x)^{\mu-1}\,dx =$$

$$= \frac{(2a)^{\nu}\,\xi^{\mu+2\nu}\,\Gamma(\nu+\frac{1}{2})}{\pi^{\frac{1}{2}}\,\Gamma(\mu+2\nu+1)}\,{}_1F_1\left(\nu+\tfrac{1}{2}; \mu+2\nu+1; -2a\xi\right) \qquad \text{Eq.(B.3)}$$

$$\mathrm{Re}\,\mu > 0\,, \quad \mathrm{Re}\,\nu > -\tfrac{1}{2}$$

from which Equation (B.2) immediately follows by setting $\nu=0$, a=1 and $\mu=1$.

The next step makes use of the Weyl integral[22)]

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21) A. Erdélyi et al., op. cit., reference 10, Vol. II, p. 197.
22) A. Erdélyi et al., op. cit., reference 10, Vol. II, p. 207.

$$[\Gamma(\mu)]^{-1}\int_{\xi}^{\infty} x^{-\nu} e^{-ax}\, I_{\nu}(ax)(x-\xi)^{\mu-1}\, dx =$$

$$= \frac{(2a)^{\nu-\mu}\,\Gamma(\frac{1}{2}-\mu+\nu)}{\pi^{\frac{1}{2}}\,\Gamma(1-\mu+2\nu)}\, {}_1F_1\left(\tfrac{1}{2}-\mu+\nu;\, 1-\mu+2\nu;\, -2a\xi\right) \qquad \text{Eq.(B.4)}$$

$$0 < \operatorname{Re}\mu < \tfrac{1}{2} + \operatorname{Re}\nu\,, \quad \operatorname{Re}(a\xi) > 0$$

which for $\mu = \nu = a = 1$ runs

$$\int_{\xi}^{\infty} x^{-1} e^{-x}\, I_1(x)\, dx = {}_1F_1\left(\tfrac{1}{2};\, 2;\, -2\xi\right) \qquad \text{Eq.(B.5)}$$

Combining this with Equation (B.2) the relation

$$N_o(1,\xi) = \xi \int_{\xi}^{\infty} x^{-1} e^{-x}\, I_1(x)\, dx \qquad \text{Eq.(B.6)}$$

is obtained.

Keeping in mind that $I_o(x)$ satisfies the differential equation

$$I_o''(x) + \frac{1}{x} I_o'(x) - I_o(x) = 0$$

and that

$$I_o'(x) = I_1(x)$$

the remainder of the proof is straightforward. The relation

$$\int_{\xi}^{\infty} x^{-1} e^{-x}\, I_1(x)\, dx = \int_{\xi}^{\infty} e^{-x}\, I_o(x)\, dx - \int_{\xi}^{\infty} e^{-x}\, I_o''(x)\, dx$$

becomes, after two partial integrations,

$$\int_{\xi}^{\infty} x^{-1} e^{-x} I_1(x)\, dx = e^{-\xi} \left\{ I_0(\xi) + I_1(\xi) \right\}$$

from which, together with Equation (B. 6), Equation (B. 1) follows.

APPENDIX C. Some properties of Meyer's G-functions.

In this Appendix only the most important properties of Meyer's G-functions will be listed. For a more complete account the reader is referred to Erdélyi et al[23] and to Meyer's original papers, a list of which is given by Erdélyi et al, loc. cit.

The function $G_{pq}^{mn}\left(x\left|\begin{matrix}a_1, \dots\dots\dots, a_p\\ b_1, \dots\dots\dots, b_q\end{matrix}\right.\right)$, sometimes written more briefly $G_{pq}^{mn}\left(x\left|\begin{matrix}a_r\\ b_s\end{matrix}\right.\right)$, is defined by

$$G_{pq}^{mn}\left(x\left|\begin{matrix}a_1, \dots, a_p\\ b_1, \dots, b_q\end{matrix}\right.\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, x^s\, ds \qquad \text{Eq.(C.1)}$$

where $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$. An empty product is interpreted as 1 and the parameters are such that no pole of $\Gamma(b_j-s)$, $j = 1, \dots, m$ coincides with any pole of $\Gamma(1-a_k+s)$, $k = 1, \dots\dots, n$. Among the several possible paths, L, of integration the following will be used in Appendix D: a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_j-s)$, $j = 1, \dots, m$ once in the negative direction. All poles of $\Gamma(1-a_k+s)$, $k = 1, \dots\dots, n$, however, lie outside this contour. The integral converges if $q \geqslant 1$ and either $p < q$, or $p = q$ and $|x| < 1$.

With this definition the integral in Equation (C.1) can be evaluated as a sum of residues and the G-function may thus be written as a power series in x.

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23) A. Erdélyi et al., op. cit., reference 18, Vol. I, Ch. 5.

Some important relations between G-functions are

$$x^{\sigma} G_{pq}^{mn}\left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right.\right) = G_{pq}^{mn}\left(x \left| \begin{matrix} a_r + \sigma \\ b_s + \sigma \end{matrix} \right.\right) \qquad \text{Eq.(C. 2)}$$

$$G_{pq}^{mn}\left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right.\right) = G_{qp}^{nm}\left(x^{-1} \left| \begin{matrix} 1 - b_s \\ 1 - a_r \end{matrix} \right.\right) \qquad \text{Eq.(C. 3)}$$

$$(1 - a_1 + b_1)\, G_{pq}^{mn}\left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right.\right) = G_{pq}^{mn}\left(x \left| \begin{matrix} a_1 - 1, a_2, \ldots, a_p \\ b_1, \ldots\ldots, b_q \end{matrix} \right.\right) +$$

$$+ G_{pq}^{mn}\left(x \left| \begin{matrix} a_1, a_2, \ldots\ldots, a_p \\ b_1 + 1, b_2, \ldots, b_q \end{matrix} \right.\right) \qquad \text{Eq.(C. 4)}$$

$$m, n \geqslant 1$$

$$(a_p - a_1)\, G_{pq}^{mn}\left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right.\right) = G_{pq}^{mn}\left(x \left| \begin{matrix} a_1 - 1, a_2, \ldots, a_p \\ b_1, \ldots\ldots, b_q \end{matrix} \right.\right) +$$

$$+ G_{pq}^{mn}\left(x \left| \begin{matrix} a_1, \ldots, a_{p-1}, a_p - 1 \\ b_1, \ldots\ldots\ldots, b_q \end{matrix} \right.\right) \qquad \text{Eq.(C. 5)}$$

$$1 \leqslant n \leqslant p - 1$$

$$x \frac{d}{dx} G_{pq}^{mn}\left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right.\right) = G_{pq}^{mn}\left(x \left| \begin{matrix} a_1 - 1, a_2, \ldots, a_p \\ b_1, \ldots\ldots, b_q \end{matrix} \right.\right) + (a_1 - 1)\, G_{pq}^{mn}\left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right.\right) \qquad \text{Eq.(C. 6)}$$

$$n \geqslant 1$$

The behaviour of $G_{pq}^{mn}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right)$ in the neighborhood of the point $x = 0$ can be obtained from Equation (C.1). One has

$$G_{pq}^{mn}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = O\left(|x|^{\beta}\right) \quad \text{as} \quad x \rightarrow 0 \qquad \text{Eq.(C.7)}$$

where $p \leqslant q$ and $\beta = \min \operatorname{Re} \; b_h$, $h = 1, 2, \ldots, m$.

APPENDIX D. Derivation of an expression for e(t) in terms of Meyer's G-functions and expansion of these into power series.

In this Appendix an expression will be derived for e(t) in terms of Meyer's G-functions. Furthermore it will be shown how these functions may be expanded into a power series in ξ , thereby permitting the computation of e(t) for small values of ξ (large times). Extensive use will be made of the properties of the G-functions listed in Appendix C.

One starts from the integral representation of g(t) derived in Section 2.2., Equation (60):

$$g(t) = C(\rho,\varphi)\,\xi^{\frac{1}{2}}\int_{\xi}^{\infty} x^{-\frac{3}{2}}(x-\xi)^{-\frac{1}{2}} e^{-x}\left\{I_0(x)+2I_1(x)\right\}dx \qquad \text{Eq.(D.1)}$$

which is split up into two integrals of the form

$$\int_{\xi}^{\infty} x^{-\frac{3}{2}}(x-\xi)^{-\frac{1}{2}} e^{-x} I_\nu(x)\,dx \qquad \nu = 0, 1.$$

This integral can be found in a table of Weyl transforms[24] and is equal to

$$\int_{\xi}^{\infty} x^{-\frac{3}{2}}(x-\xi)^{-\frac{1}{2}} e^{-x} I_\nu(x)\,dx = 2^{\frac{3}{2}}\xi^{\frac{1}{2}}\, G_{23}^{21}\left(2\xi \,\middle|\, \begin{matrix} -1 & 0 & \\ -\frac{1}{2} & \nu-\frac{3}{2} & -\nu-\frac{3}{2} \end{matrix}\right) \qquad \text{Eq.(D.2)}$$

Substitution into Equation (D.1) gives

$$g(t) = 2^{\frac{3}{2}} C(\rho,\varphi)\,\xi\left[G_{23}^{21}\left(2\xi \,\middle|\, \begin{matrix} -1 & 0 & \\ -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \end{matrix}\right) + 2\,G_{23}^{21}\left(2\xi \,\middle|\, \begin{matrix} -1 & 0 & \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \end{matrix}\right)\right] \qquad \text{Eq.(D.3)}$$

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24) A. Erdélyi et al., op. cit., reference 10, Vol. II, p. 207.

or, using Equation (C.2)

$$g(t) = 2^{\frac{1}{2}}\, C(\rho,\varphi)\left[G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} 0 & 1 & \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{matrix}\right) + 2\, G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} 0 & 1 & \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \end{matrix}\right)\right] \qquad \text{Eq. (D.4)}$$

From this it follows for e(t), making use of Equations (32) and (C.6)

$$e(t) = \frac{d\,g(t)}{dt} = -\frac{\xi}{t}\,\frac{d\,g(t)}{d\,\xi} =$$

$$= -\frac{8\sqrt{2}}{\mu_0 \sigma \rho^2}\, C(\rho,\varphi)\,\xi \left[G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} -1 & 1 & \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{matrix}\right) - G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} 0 & 1 & \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{matrix}\right) + \right.$$

$$\left. + 2\, G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} -1 & 1 & \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \end{matrix}\right) - 2\, G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} 0 & 1 & \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \end{matrix}\right)\right] \qquad \text{Eq. (D.5)}$$

This can be simplified somewhat with the aid of Equation (C.5), giving

$$e(t) = \frac{8\sqrt{2}}{\mu_0 \sigma \rho^2}\, C(\rho,\varphi)\,\xi \left[G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} 0 & 0 & \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{matrix}\right) + 2\, G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \end{matrix}\right)\right] \qquad \text{Eq. (D.6)}$$

which, again using Equation (C.2), may be written as

$$e(t) = \frac{4\sqrt{2}}{\mu_0 \sigma \rho^2}\, C(\rho,\varphi) \left[G^{21}_{23}\left(2\xi \,\middle|\, \begin{matrix} 1 & 1 & \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) + 2\, G\left(2\xi \,\middle|\, \begin{matrix} 1 & 1 & \\ \frac{3}{2} & \frac{3}{2} & -\frac{1}{2} \end{matrix}\right)\right] \qquad \text{Eq. (D.7)}$$

By writing out $C(\rho,\varphi)$ Equation (67) is obtained.

From Equation (D.7) it is now possible to extract information concerning the behaviour of e(t) as $t \longrightarrow \infty$. According to Equation (C.7).

$$G_{23}^{21}\left(2\xi \middle| \begin{matrix} 1 & 1 & \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = O(\xi^{\frac{1}{2}})$$

and Eq.(D.8)

$$G_{23}^{21}\left(2\xi \middle| \begin{matrix} 1 & 1 & \\ \frac{3}{2} & \frac{3}{2} & -\frac{1}{2} \end{matrix}\right) = O(\xi^{\frac{3}{2}})$$

as $\xi \longrightarrow 0$. Therefore e(t) approaches zero like $t^{-\frac{1}{2}}$ as $t \longrightarrow \infty$.

Equations (D.7) and (67) are not suited for numerical computation of e(t). For short times t the asymptotic expansion Equation (64) may be used for this purpose, but for large values of t (small ξ) it is necessary to expand Equation (D.7) into a series of ascending powers of ξ. This is done by replacing the G-functions by their integral representation Equation (C.1) and evaluating each contour integral as a series of residues.

The integral representation of the first term in Equation (D.7) is, (temporarily replacing 2ξ by y)

$$G_{23}^{21}\left(y \middle| \begin{matrix} 1 & 1 & \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\Gamma(\frac{3}{2}-s)\,\Gamma(\frac{1}{2}-s)\,\Gamma(s)}{\Gamma(\frac{1}{2}+s)\,\Gamma(1-s)}\, y^s\, ds \qquad \text{Eq.(D.9)}$$

The poles and the integration path L are sketched in Figure 14. There is a simple pole in $s = \frac{1}{2}$ and two-fold poles in $s = {}^3/2, {}^5/2, {}^7/2, \ldots\ldots$ The residue $k_{\frac{1}{2}}$ in the point $s = \frac{1}{2}$ is found from

$$k_{\frac{1}{2}} = \lim_{s \to \frac{1}{2}} (s - \tfrac{1}{2}) \frac{\Gamma(\frac{3}{2}-s)\,\Gamma(\frac{1}{2}-s)\,\Gamma(s)}{\Gamma(\frac{1}{2}+s)\,\Gamma(1-s)}\, y^s = -y^{\frac{1}{2}} \qquad \text{Eq.(D.10)}$$

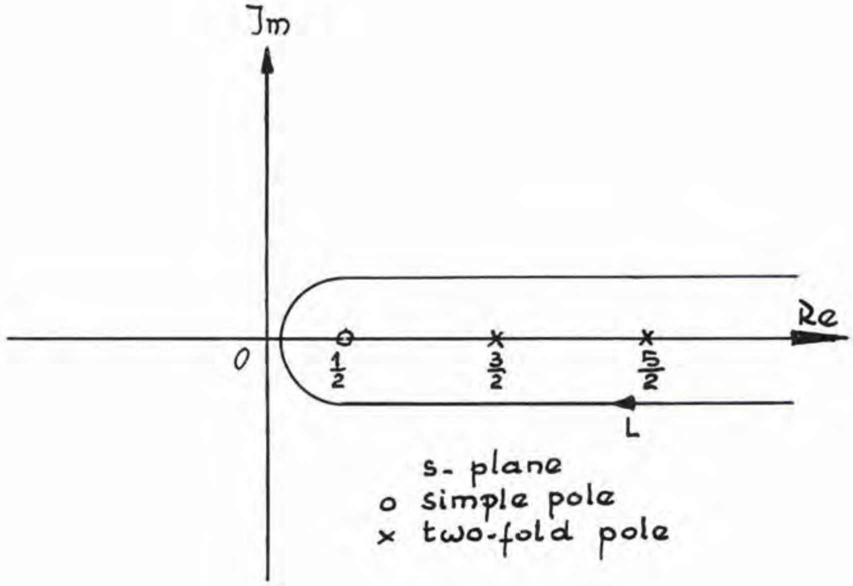

Fig. 14. The contour used for the evaluation of the integral in Eq. (D. 9)

while the residues $k_{n+\frac{1}{2}}$, $n = 1, 2, 3 \ldots\ldots$ are obtained from

$$k_{n+\frac{1}{2}} = \lim_{s \to n+\frac{1}{2}} \frac{d}{ds} \left\{ \left(s-n-\tfrac{1}{2}\right)^2 \frac{\Gamma(\frac{3}{2}-s)\,\Gamma(\frac{1}{2}-s)\,\Gamma(s)}{\Gamma(\frac{1}{2}+s)\,\Gamma(1-s)}\, y^s \right\}$$

$$= \frac{(-)^{n-1}}{\pi} \frac{\Gamma^2(n+\frac{1}{2})}{\Gamma^3(n+1)}\, y^{n+\frac{1}{2}} \left[1 + n \left\{ \ln y + 2\psi(n+\tfrac{1}{2}) - 3\psi(n+1) \right\} \right] \qquad \text{Eq. (D. 11)}$$

$$n = 1, 2, 3, \ldots\ldots$$

where the function ψ (z) is the logarithmic derivative of the gamma function,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

and extensive use has been made of the properties of gamma and ψ -functions. Since the contour integral in Equation (D. 9) equals minus $2\pi i$ times the sum of the residues (the minus sign stemming from the fact that the contour is traversed in the negative direction) one obtains

$$G_{23}^{21}\left(y\Big|_{\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}}^{1\ \ 1}\right) = -\sum_{n=0}^{\infty} k_{n+\frac{1}{2}} \qquad \text{Eq. (D. 12)}$$

$$= y^{\frac{1}{2}}\left[1+\frac{1}{\pi}\sum_{n=1}^{\infty}(-)^n \frac{\Gamma^2(n+\frac{1}{2})}{\Gamma^3(n+1)} y^n\left\{1+n\left[\ln y+2\psi(n+\tfrac{1}{2})-3\psi(n+1)\right]\right\}\right]$$

from which it follows immediately that for small y

$$G_{23}^{21}\left(y\Big|_{\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}}^{1\ \ 1}\right) \approx y^{\frac{1}{2}}$$

which is in agreement with the first of Equations (D. 8). In an analogous way it can be shown that

$$G_{23}^{21}\left(y\Big|_{\frac{3}{2}\ \frac{3}{2}\ -\frac{1}{2}}^{1\ \ 1}\right) = \frac{y^{\frac{1}{2}}}{\pi}\sum_{n=1}^{\infty}(-)^{n-1}\frac{\Gamma^2(n+\frac{1}{2})}{\Gamma^2(n)\Gamma(n+2)} y^n\left\{\ln y+2\psi(n+\tfrac{1}{2})-2\psi(n)-3\psi(n+2)\right\} \qquad \text{Eq. (D. 13)}$$

Substituting Equations (D. 12) and (D. 13) into Equation (D. 7) one obtains for e(t)

$$e(t) = \frac{8}{\mu_0\sigma\rho^2} C(\rho,\varphi)\xi^{\frac{1}{2}}\left[1+\frac{1}{\pi}\sum_{n=1}^{\infty}(-)^n\frac{\Gamma^2(n+\frac{1}{2})}{\Gamma^3(n+1)}(2\xi)^n\cdot\right.$$

$$\left.\cdot\left\{1+n\left[\ln(2\xi)+2\psi(n+\tfrac{1}{2})-3\psi(n+1)\right]-\frac{2n^2}{n+1}\left[\ln(2\xi)+2\psi(n+\tfrac{1}{2})-2\psi(n)-3\psi(n+2)\right]\right\}\right] =$$

$$= \frac{8}{\mu_0\sigma\rho^2} C(\rho,\varphi)\xi^{\frac{1}{2}}\left[1+\frac{1}{\pi}\sum_{n=1}^{\infty}(-)^{n-1}\frac{\Gamma^2(n+\frac{1}{2})}{\Gamma^2(n+1)\Gamma(n+2)}(2\xi)^n\cdot\right.$$

$$\left.\cdot\left\{\frac{n^2+2n-1}{n+1}+n(n-1)\left[\ln(2\xi)+2\psi(n+\tfrac{1}{2})-3\psi(n+1)\right]\right\}\right] \qquad \text{Eq. (D. 14)}$$

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