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**A method for determining absolute  
velocities from hydrographic data**

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Ralph R. Goodman  
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velocities from hydrographic data**

G. Peggion

**Abstract:** This study describes a procedure for deriving the characteristics of large-scale climatic currents from hydrographic data of a variable depth region of the ocean. The model equations are written on a  $\beta$ -plane, using the geostrophic, hydrostatic, and Boussinesq approximations. The effects of surface and bottom boundary layers, and turbulent transfers of mass and momentum are neglected.

The model is formulated such that the velocities, which are the solutions of the thermal wind equations, are referred to the (unknown) bottom velocities. Within the assigned dynamical constraints, a differential equation for the bottom pressure associated with the bottom geostrophic velocities can be derived. The equation would hold exactly if the model equations exactly described the motion, or if the hydrographic data did not contain errors and noise. The misfit, which is assumed never to be zero, is an additional unknown component of the problem. To handle the indeterminacy of the problem, the absolute velocities are chosen to minimize the variance of the misfit over the whole region of interest. The method is illustrated by applications.

**Keywords:** absolute velocities ◦ hydrographic data ◦ inverse methods ◦ thermal wind equations ◦ variational principle

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## 1. Introduction

A classical problem in physical oceanography is how to calculate velocity field profiles from the observed density distribution. The starting point of this problem is to describe the large-scale climatic currents averaged over a long-term interval in some domain  $V$  of the open sea by disregarding the surface and bottom boundary layers and the effects of turbulent transfers of mass and momentum.

The model equations are usually written on a  $\beta$ -plane using geostrophy, and hydrostatic, and Boussinesq approximations. As is well known, the use of geostrophic balances defines only the vertical shear of the horizontal velocity, which upon integration leaves the geostrophic velocity undetermined by an integration constant that generally varies from one location to another. The indeterminacy of the problem has been traditionally removed by making a somewhat arbitrary choice of a level of no-motion, or by reference to direct observation of the velocity field at some depth. Stommel and Schott (1977) computed the absolute velocities from the observed density data alone, by further assuming immiscibility of the density stratification and using a simple linear  $\beta$ -plane potential vorticity conservation law. Wunsch (1978) developed a method which considered a limited number of isotherms as material surfaces. Within the layers corresponding to the material surfaces, an arbitrary number of conservation laws are assigned, and the solution is one of minimum energy. Davis (1978) showed the differences and similarities between these two methods, and illustrated how errors in the data might be reflected in the solution. Stommel and Veronis (1981) kept the multilayer formulation for a variational inverse method in which the 'barotropic' energy is minimized subject to constraints imposed on the transport of the layers. Needler (1985) and Killworth (1979) showed that the velocity field can be determined at any given point by using no information other than that on the potential vorticity and Bernoulli function on isotherms and their changes from one surface to another. The method assumes smooth field profiles, which is not always the case. Provost and Salmon (1986) developed a fully three-dimensional variational method, requiring solutions with minimum energy (or minimum roughness) and subjected to dynamical constraints such as geostrophy, hydrostatic balance, and Boussinesq approximations. This is a very elegant mathematical method, but perhaps it places too much emphasis on the 'smoothing' functional.

It is useful to classify the methods which have been mentioned and other methods into three groups:

- (i) *Global methods* (Stommel and Veronis, Provost and Salmon): the constraints are simultaneously imposed over the whole domain of interest, thus preserving the large-scale nature of the solutions.

- (ii) *Column methods* (Stommel and Schott): the integration constant from the thermal wind equations is computed for each water column, separately.
- (iii) *Local methods* (Needler, Killworth): the velocity field is computed locally, point by point.

Each group represents the relation between the scale definition and insensitivity to noise of the 'smooth' field solution. Once the mathematical framework has been defined, the selection of a global method implicitly assumes that the smooth field has a large scale in comparison with the station spacing, and that the small-scale noise in the observations is uncorrelated between stations. The selection of a column method is equivalent to assuming that in the given model the scale of the solution is comparable to that of the station grid (Davis, 1978). And finally, the selection of a local method considers virtually noise-free hydrographic data.

In this study we describe a global method for deriving absolute velocities from hydrographic data of a variable-depth region of the ocean. The velocities (solutions of the thermal wind equations) are referred to the (unknown) bottom velocities. By assuming that the density distribution is known and conserved inside the domain of interest, a differential equation for the pressure at the bottom relative to the geostrophic bottom velocities is derived. The equation would hold exactly if the model equations exactly described the motion, or if the hydrographic data did not contain errors and noise. The misfit, which is assumed never to be zero, is an additional unknown component of the problem. Thus we define as our solution the bottom velocities that minimize the misfit (in the sense of an arbitrarily chosen norm), consistent with the data and the assigned dynamics.



## 2. The model equations

Consider the usual model formulation on a  $\beta$ -plane with a cartesian coordinate system  $x, y, z$  chosen such that in the northern hemisphere the  $x$ -coordinate increases eastwards, the  $y$ -coordinate northwards, and the vertical coordinate  $z$  is zero at a bottom reference level and increases upwards. Assume that the ocean is non-diffusive, hydrostatic and incompressible, and the flow geostrophic and inviscid. Thus, the model equations are

$$-fv = -p_x / \rho_0, \quad (2.1a)$$

$$fu = -p_y / \rho_0, \quad (2.1b)$$

$$0 = -p_z - g\rho, \quad (2.1c)$$

$$u_x + v_y + w_z = 0, \quad (2.1d)$$

$$u\rho_x + v\rho_y + w\rho_z = 0, \quad (2.1e)$$

$$f = f_0 + \beta y. \quad (2.1f)$$

(See Appendix A for a complete dimensional analysis of the model equations.)

The subscripts  $x, y, z$  denote partial differentiations, and the variables  $u, v, w$  are the components of the eastward, poleward, and vertical velocity, respectively. The variable  $p$  is the hydrostatic pressure associated with the density distribution  $\rho$ ,  $\rho_0$  is a density constant of reference,  $g$  is the gravitational acceleration, and  $f$  the Coriolis parameter written in the usual  $\beta$ -plane approximation. The bottom of the ocean is taken to be at  $z = h(x, y)$ .

In the presence of material boundaries there are no fluxes across the boundaries, which is the condition most consistent with the model formulation which neglects boundary layer dynamics. Thus, in order to reduce the distortions due to the model approximations as much as possible, we assume that Eqs. (2.1) are satisfied in an open region of the ocean, say the domain  $V : \{|x| \leq x_0, |y| \leq y_0, h \leq z \leq H\}$ . According to the usual formulation of the inverse model, the density distribution is known inside the domain  $V$ . Pressure and velocity fields are the unknowns of the problem.

Let the pressure  $p$  be decomposed as follows:

$$p(x, y, z) = p(z = h) - g \int_h^z \rho dz = p_r + p_b - g \int_h^z \rho dz, \quad (2.2)$$

where  $p(z = h) = p(x, y, h(x, y))$  is the pressure at the bottom decomposed into the pressure  $p_r$  (related to a state of rest) and the pressure  $p_b$  (associated with the geostrophic bottom velocities).

According to the decomposition (2.2), let the velocity field be decomposed as follows:

$$(u, v, w) = (u' + u_b, v' + v_b, w' + w_b), \quad (2.3a)$$

such that

$$\begin{aligned} -fv' &= \frac{g}{\rho_0} \int_h^z \rho_x dz, & fu' &= \frac{g}{\rho_0} \int_h^z \rho_y dz, \\ fw_z &= \beta v' - Y, \end{aligned} \quad (2.3b)$$

$$\begin{aligned} -fv_b &= -\frac{1}{\rho_0} (p_b)_x, & fu_b &= -\frac{1}{\rho_0} (p_b)_y, \\ fw_{bz} &= \beta v_b + Y, \end{aligned} \quad (2.3c)$$

where  $Y = (g/\rho_0)(\rho_x(z = h)h_y - \rho_y(z = h)h_x)$ . In deriving (2.3) we have used the hydrostatic approximation and the relationship  $(p(z = h))_x = p_x(z = h) + p_z(z = h)h_x$ , and similar composite derivatives.

Thus the velocities and  $u'$  and  $v'$  are the solutions of the thermal wind equations

$$u'_z = \frac{g}{\rho_0 f_0} \rho_y, \quad (2.4a)$$

$$v'_z = -\frac{g}{\rho_0 f_0} \rho_x, \quad (2.4b)$$

referred to the (unknown) bottom velocities ( $u_b$  and  $v_b$ ).

At this point it would have been easy to adopt the terms baroclinic and barotropic for the velocity vectors  $u'$  and  $u_b$ , respectively, but much confusion exists in the literature with regard to these terms. Thus, we have considered the usual definition, whereby the barotropic component is associated with the sea surface displacement and is constant throughout the water column, and the baroclinic component is associated with the solution of (2.4) in which the sea level is taken as the level of reference (LeBlond and Mysak, 1978). Thus, the following relationships apply:

$$\mathbf{u}_{\text{baroclinic}} = \mathbf{u}' - \mathbf{u}'(z = H),$$

$$\mathbf{u}_{\text{barotropic}} = \mathbf{u}_b + \mathbf{u}'(z = H).$$

The mass conservation equation (2.1e) gives the coupling mechanism between the two velocity fields:

$$(u_b \rho_x + v_b \rho_y + w_b \rho_z) = (u' \rho_x + v' \rho_y + w' \rho_z). \quad (2.5)$$

To close the problem we must now specify the vertical velocities  $w'$  and  $w_b$ . The usual boundary condition of no flow through the bottom leads to

$$w = u h_x + v h_y = u_b h_x + v_b h_y, \quad \text{at } z = h. \quad (2.6a)$$

Since the boundary condition (2.6a) involves only the bottom velocities, a natural choice for the boundary conditions on  $w'$  and  $w_b$  separately is

$$w' = 0, \quad w_b = u_b h_x + v_b h_y, \quad \text{at } z = h. \quad (2.6b)$$

The correct formulation of (2.6b) would be the introduction of a new (and unknown) parameter  $W$ , such that

$$w' = W, \quad w_b = u_b h_x + v_b h_y - W, \quad \text{at } z = h.$$

However, since Eq. (2.1e) is linear in  $w$ , the model is independent of  $W$ , and we can set  $W = 0$ . Similarly, the term  $Y$  in (2.3) may be neglected, and the vertical velocities are specified as follows:

$$w' = \frac{\beta}{f_0} \int_h^z v', \quad (2.7a)$$

$$w_b = u_b h_x + v_b h_y + \frac{\beta}{f_0} (z - h) v_b. \quad (2.7b)$$

Finally, substitution of (2.3) and (2.7) into (2.5) gives a differential equation for the pressure  $p_b$ , i.e.

$$a p_{bx} + b p_{by} = F, \quad (2.8a)$$

where

$$a = \rho_y + h_y \rho_z + \frac{\beta}{f_0} (z - h) \rho_z, \quad b = -(\rho_x + h_x \rho_z),$$

$$F = -f \rho_0 (u' \rho_x + v' \rho_y + w' \rho_z). \quad (2.8b)$$

### 3. The variational principle

#### 3.1. FORMULATION OF THE VARIATIONAL METHOD

Since we have assumed that density is known from direct measurements in a region  $V$  of the ocean, Eq. (2.8) can be solved for the unknown pressure at the bottom,  $p_b$ . However, the coefficients of the equation may contain noise in the data and errors due to the fact that the initial model formulation (2.1) ignores terms that might be dynamically important. Thus a more realistic formulation is to consider the equation

$$ap_{bx} + bp_{by} - F = R, \quad (3.1)$$

where  $R$  represents a residual (unknown) function.

The differential nature of Eq. (3.1) requires a specification of boundary conditions. To retain a simple formulation for the method, let us assume that the velocity across the lateral boundary is zero at the bottom:

$$p_b = 0, \quad \text{on } \partial D, \quad (3.2)$$

where  $D : \{|x| \leq x_0, |y| \leq y_0\}$ .

To preserve the nature of the problem, which relates to computing large-scale climatic currents averaged over a long-term period, we define as the solution of the problem (3.1)-(3.2) the function  $p_b$  that minimizes the variance of the misfit over the whole region of interest, i.e. we define  $p_b$  as the solution of the variational problem

$$J(p) = \left( \int_V (ap_x + bp_y - F)^2 dV \right)^{1/2} = \min \quad (3.3)$$

for all  $p$ , such that  $p = 0$  on  $\partial D$ .

Integration over depth of (3.3) leads to

$$J(p) = \left( \int_D \langle a^2 \rangle p_x^2 + \langle b^2 \rangle p_y^2 + 2\langle ab \rangle p_x p_y - 2\langle aF \rangle p_x - 2\langle bF \rangle p_y + \langle F^2 \rangle dx dy \right)^{1/2}, \quad (3.4)$$

where  $\langle \dots \rangle = \int_h^H \dots dz$ .

Thus for all  $\hat{h} \in C_0^\infty(D)$ , the functional  $J_\epsilon(p_b + \epsilon \hat{h})$  has a minimum for  $\epsilon = 0$ , i.e.

$$\left( \frac{\partial J_\epsilon}{\partial \epsilon} \right)_{\epsilon=0} = 0$$

(Gelfand and Fomin, 1963). Thus it follows that

$$\int (\langle a^2 \rangle p_{bx} + \langle b^2 \rangle p_{by} - \langle aF \rangle) \hat{h}_x + (\langle b^2 \rangle p_{bx} + \langle ab \rangle p_{by} - \langle bF \rangle) \hat{h}_y = 0, \\ \forall \hat{h} \in C_0^\infty(D).$$

Integration by parts and the boundary conditions assigned to  $\hat{h}$  imply

$$\int_D L(p_b) \hat{h} \, dx \, dy = 0,$$

where

$$L(p) = (\langle a^2 \rangle p_x)_x + (\langle b^2 \rangle p_y)_y + (\langle ab \rangle p_x)_y + (\langle ab \rangle p_y)_x - (\langle aF \rangle_x + \langle bF \rangle_y). \quad (3.5)$$

From the arbitrariness of the function  $\hat{h}$  it follows that the bottom pressure,  $p_b$  must satisfy the differential problem

$$L(p) = 0, \\ p = 0, \quad \text{on } \partial D. \quad (3.6)$$

We observe that the operator  $L$  is elliptic. This follows from the Holder inequality applied to the characteristic equation associated with the operator  $L$ , which becomes singular if and only if the coefficients  $a$  and  $b$  of (2.8b) are proportional. We recall that the assumption  $p_b = 0$  on  $\partial D$  does not alter the mathematical formulation of the variational principle. Other boundary conditions, such as those involving derivatives of  $b$ , must be carefully treated but are still mathematically acceptable. With respect to the dependence of the solution on the boundary conditions, the elliptic nature of (3.6) guarantees that regularity properties of the solution and its derivatives hold from appropriate regularity properties of the data and domain (Miranda, 1970).

### 3.2. A GENERALIZATION OF THE METHOD

The procedure illustrated in Subsect. 3.1 assumes a knowledge of the density distribution alone. However, the density distribution is usually computed from direct measurements of the salinity  $S$  and temperature  $T$  through an equation of state of the form

$$\rho = \rho(S, T). \quad (3.7)$$

Thus the use of density alone reduces the information contained in a hydrographic data set (Olbers et al., 1985). Furthermore, tracer and other seawater property distributions are becoming increasingly available and can be used as additional

sources of information for the solution of this inverse problem (Fiadeiro and Veronis, 1984; Wunsch, 1985). Thus let us assume that  $N$  characteristic properties  $C_k$  (such as tracer and constituent distributions) are known inside the domain of interest. Consistent with the assumptions of the model formulation (2.1e), the properties  $C_k$  satisfy the conservation equations

$$u(C_k)_x + v(C_k)_y + w(C_k)_z = \lambda_k C_k + Q_k, \quad k = 1, \dots, N, \quad (3.8)$$

where  $\lambda_k$  are the eventual decay timescales, and  $Q_k$  are the sources of the corresponding constituents.

Similarly to the decompositions (2.5) and (2.8), Eqs. (3.8) are written as follows:

$$a_k p_{bx} + b_k p_{by} - F_k = R_k, \quad k = 1, \dots, N, \quad (3.9)$$

where  $R_k$  represents the unknown residual function. Therefore, if more than one field distribution is known inside the domain of interest we can define  $p_b$  (the solution of the variational problem):

$$J^*(p) = \left( \sum_{k=1}^N \Theta_k \int_V \varphi_k (a_k p_x + b_k p_y - F_k)^2 dV \right)^{1/2} = \min \quad (3.10a)$$

in which  $\Theta_k$  are constants, and  $\varphi_k$  are weight functions assigned *a priori*. Without loss of generality, we assume

$$\sum_{k=1}^N \Theta_k = 1, \quad 0 \leq \varphi_k \leq 1. \quad (3.10b)$$

Although the weights  $\Theta_k$  and  $\varphi_k$  are not essential to the development of the variational principle, they may be of importance when assessing the quality of the solution. The constants  $\Theta_k$  allow us to dynamically rank one constituent equation above the others: larger values of  $\Theta_k$  should be assigned when smaller values of the corresponding misfit are expected. On the other hand, the weight functions  $\varphi_k$  are the mechanisms for correcting and adjusting each data set. Smaller values of the functions  $\varphi_k$  correspond to less reliable data, and zero values are assigned when the data are not available.

Repeating the procedure of the previous section, it follows that the bottom pressure must satisfy the differential problem:

$$L^*(p) = (a^* p_x)_x + (b^* p_y)_y + (c^* p_x)_y + (c^* p_y)_x - F^*, \quad (3.11a)$$

where

$$a^* = \sum_{k=1}^N \Theta_k \langle \varphi_k a_k^2 \rangle, \quad b^* = \sum_{k=1}^N \Theta_k \langle \varphi_k b_k^2 \rangle, \quad c^* = \sum_{k=1}^N \Theta_k \langle \varphi_k a_k b_k \rangle,$$

$$F^x = \sum_{k=1}^N \Theta_k (\langle \varphi_k a_k F_k \rangle_x + \langle \varphi_k b_k F_k \rangle_y). \quad (3.11b)$$

## 4. Applications

### 4.1. THE NUMERICAL EXPERIMENTS.

Here we apply our numerical procedure to show the validity of the method through applications. Numerical experiments are performed on a  $11 \times 11 \times 11$ -point mesh where the salinity and temperature distributions are given. Density is computed from a simplified linear equation of state (Bryan and Cox, 1972). An exponential-in- $x$  and linear-in- $y$  topography is included, as depicted in Fig. 1. Because of the theoretical nature of the data input, the parameters  $\Theta_k$  and the functions  $\varphi_k$  of (3.10) are set equal and constant. Values of the other parameters and constants used in the simulations are given in Table 1. Although the model is not undetermined with the constituents stratified and horizontally homogeneous (i.e. the solution is of no-motion), a small linear poleward gradient is assigned to the data input to emphasize the role of the  $u$ -velocity. As Fig. 2 shows, the density distribution simulates a front of warm, salty water embedded in a region of cold, fresh water. These features are an approximation of the Atlantic inflow to the Norwegian Sea.

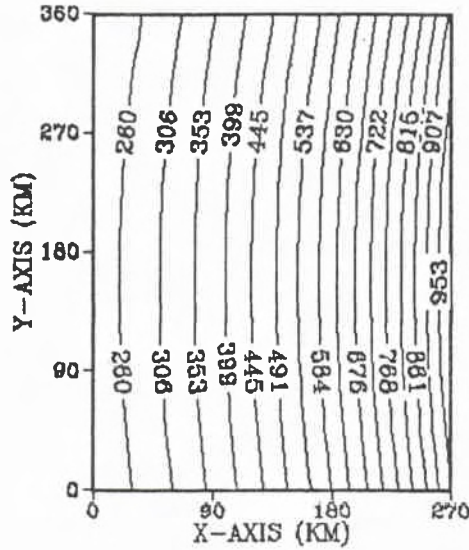


Fig. 1: Height above the reference level  $z = 0$  of the topography profile that was used in the numerical simulations. [Values are in m.]



TABLE 1  
Values of constants and parameters used in the numerical experiments

$\rho_0$	$1 \text{ g cm}^{-3}$	Reference density
$f_0$	$10^{-4} \text{ s}^{-1}$	Coriolis parameter
$\beta$	$10^{-13} \text{ cm}^{-1} \text{ s}^{-1}$	Beta parameter
$g$	$981 \text{ cm}^2 \text{ s}^{-1}$	Gravitational acceleration
$x_0$	$3 \times 10^7 \text{ cm}$	Width of the domain
$y_0$	$4 \times 10^7 \text{ cm}$	Length of the domain
$H$	$1.8 \times 10^5 \text{ cm}$	Maximum depth
$T_p$	$1.2 \times 10^5 \text{ cm}$	Maximum topography height
$r_s$	$0.25 \times 10^{-8} \text{ ppt cm}^{-1}$	Poleward salinity gradient
$r_t$	$0.5 \times 10^{-7} \text{ }^\circ\text{C cm}^{-1}$	Poleward temperature gradient
$E_s$	$0.05 \text{ ppt}$	Noise amplitude for salinity
$E_t$	$0.3 \text{ }^\circ\text{C}$	Noise amplitude for temperature
$\Delta x$	$15 \times 10^5 \text{ cm}$	$x$ -spacing
$\Delta y$	$20 \times 10^5 \text{ cm}$	$y$ -spacing
$\Delta z$	$10^5 \text{ cm}$	$z$ -spacing

(i) *Smooth data fields.* Let us first analyse the solution obtained by requesting the simultaneous minimum misfit for salinity, temperature, and density distributions; henceforth, this will be referred to as the General Case. Fig. 3 illustrates the central east–west section of the velocity components. The flow indicates the presence of a strong poleward jet corresponding to the warm-water core, and a second, weaker poleward jet further to the west. Both currents are located in regions where the density  $x$ -derivative has local maxima.

Fig. 4 represents the solution,  $\eta$ , of the differential problem:

$$\begin{aligned} \Delta \eta &= (v_x - u_y)_{z=H}, \\ \frac{\partial \eta}{\partial \mathbf{n}} &= -\mathbf{u} \cdot \mathbf{r}, \quad \text{on } \partial D, \end{aligned} \quad (4.1)$$

where  $\mathbf{n}$  and  $\mathbf{r}$  are respectively the directions normal and parallel to the boundary. If the flow were not three-dimensional and not referred on a  $\beta$ -plane, the function  $\eta$  should have been the usual stream function. However, since the vertical velocity is an order of magnitude smaller than the horizontal velocity scale, we consider Fig. 4 to be representative of the barotropic flow, and thus we retain the term stream function for the variable  $\eta$ .

Fig. 5 represents the amount of energy generated by the solution of the variational principle. We observe two regions of local maximum energy input: one is located at the highest topography position, the other lies between the two poleward jets. Although the former energy input is small in comparison to the energy contained

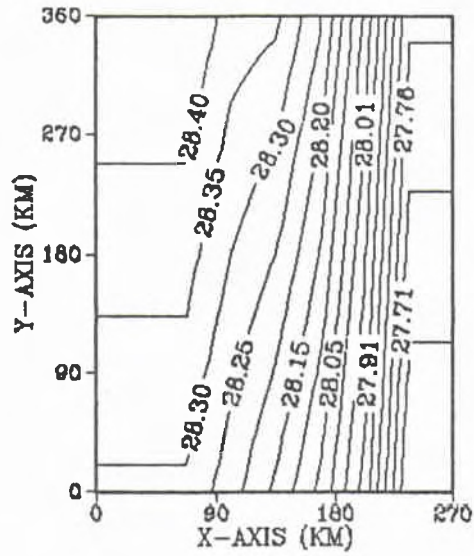


Fig. 2a:  $\sigma$ - $t$  distribution for the upper level.

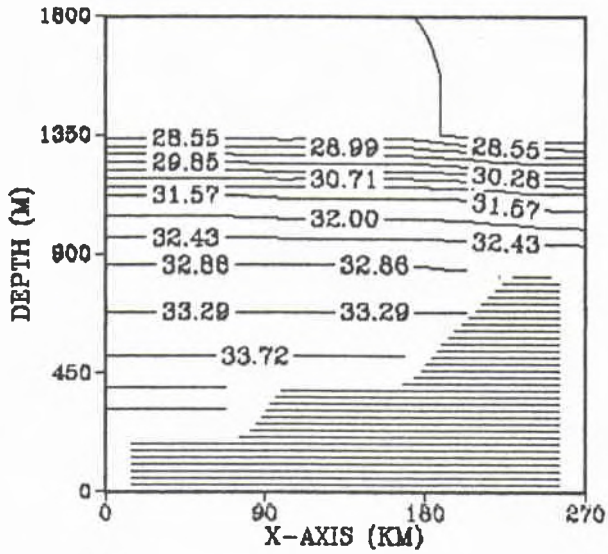


Fig. 2b: Central east-west section of  $\sigma$ - $t$  distribution.

Fig. 2: Density and constituent distributions used in the numerical simulations.

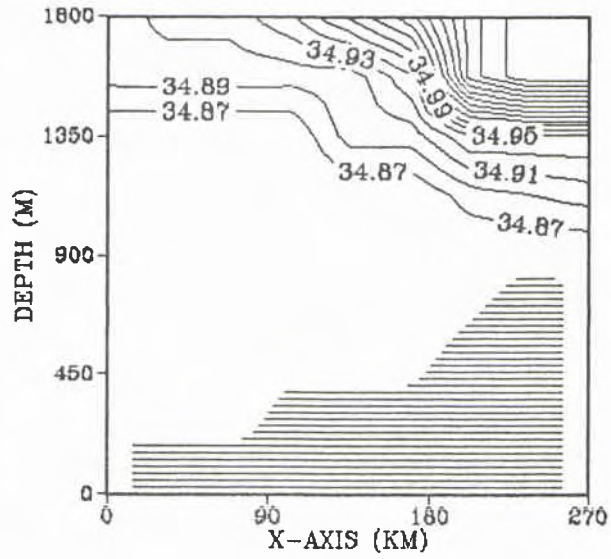


Fig. 2c: Central east-west section of the salinity. [Values are in ppt.]

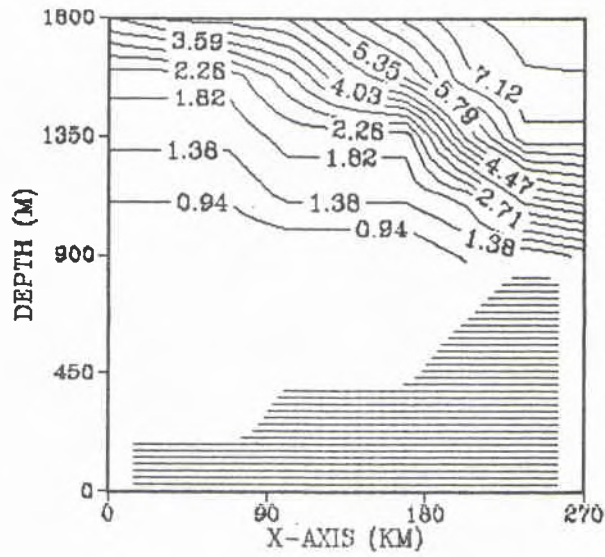


Fig. 2d: Central east-west section of the temperature. [Values are in °C.]

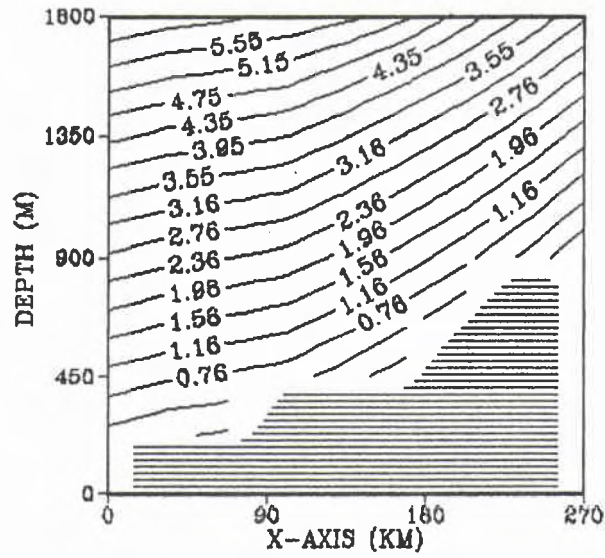


Fig. 3a:  $u$ -velocity. [Values are in  $\text{cm s}^{-1}$ .]

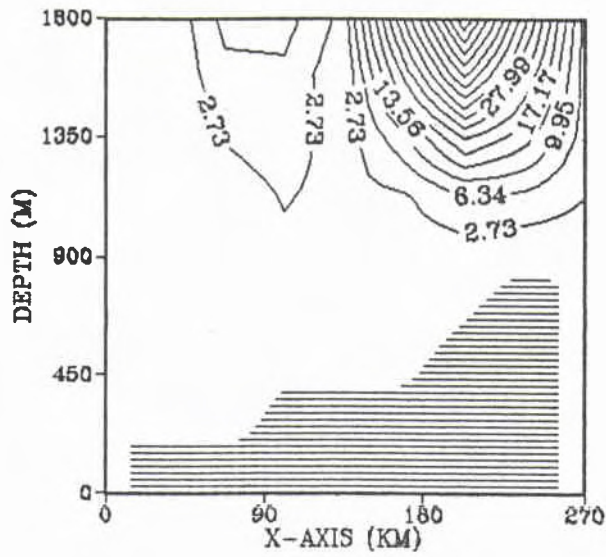


Fig. 3b:  $v$ -velocity. [Values are in  $\text{cm s}^{-1}$ .]

Fig. 3: Central east-west section of the velocity field.

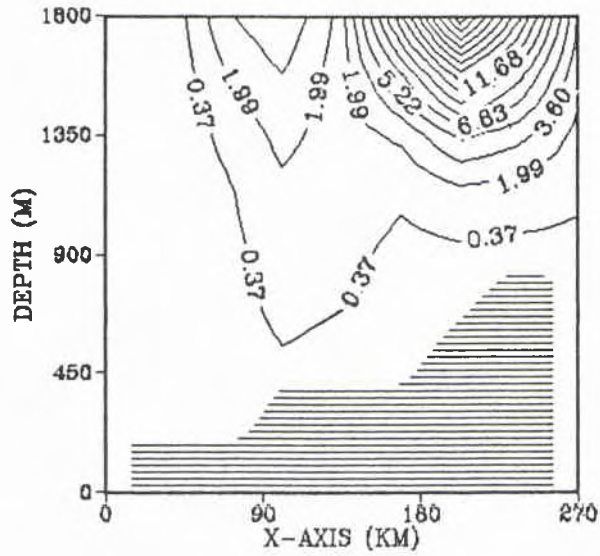


Fig. 3c:  $w$ -velocity (scaled by  $10^4$ ). [Values are in  $\text{cm s}^{-1}$ .]

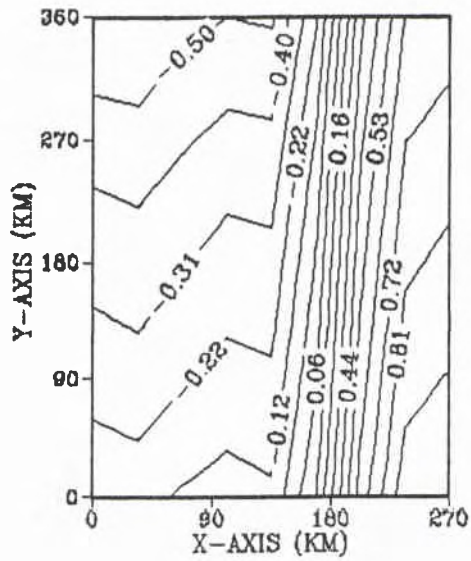


Fig. 4: Stream function for the upper level, normalized with respect to its maximum value.

in the whole water column, it is generally sufficient to conserve transport over the topography variations. We have estimated the residual of the mass conservation equation, i.e. the function  $\text{Tr}$ , such that

$$\text{Tr} = \frac{\langle u \rangle_x + \langle v \rangle_y + w(z = H)}{\langle u \rangle_x + \langle v \rangle_y}, \quad (4.2)$$

and it has been observed that up to 96% of the total horizontal transport is conserved. On the other hand, the latter local maximum of energy input is a consequence of adopting a ‘global method’: the solution tries to connect features of the motion over the maximum length scale associated with the density field.

Independent of the pattern of the solution, the energy generated by the solution of the variational principle would always be much less than the total kinetic energy of the system. In order to clarify this result, an *a priori* estimate of the bottom velocity is derived in Appendix B. The predictions and the results of the numerical simulations are compared in Table 2. For the given density distribution and domain geometry, the following values were considered: length scale  $L = 90$  km (defined as the extension of the warm front), total depth  $H = 1800$  m, density variation  $(\Delta\rho)/\rho_0 = 0.5 \times 10^{-3}$ , and a Brunt-Väisälä frequency  $N = 10^{-2} \text{ s}^{-1}$ . These values imply a planetary vorticity factor  $\beta_0 = 0.9 \times 10^{-2}$ , a stratification parameter  $s = 4.8$ , and a topographic parameter  $\lambda = 0.2$  (see Appendices A and B for definitions of these terms). As Table 2 confirms, the predictions are in good agreement with the numerical computations. Indeed the estimate from the analytical model does *not* include an application of the variational principle.

TABLE 2

Comparison between predictions and numerical computations for the General Case<sup>1</sup>

	Prediction	Max. computed values
$U$ [ = $gH\Delta\rho/f_0L\rho_0$ ]	$0.98 \times 10^2$	$0.61 \times 10^2$
$\epsilon$ [ = $U/(f_0L)$ ]	$10^{-1}$	$0.6 \times 10^{-1}$
$\Xi$ [ = $s\beta_0/\epsilon$ ]	0.4	0.64
$\Lambda$ [ = $s\lambda/\epsilon$ ]	9.6	15.1
$\mu$	$\ll 1$	$0.22 \times 10^{-9}$
$U_b$ [ = $UO(\Xi, \mu)/O(1, \Xi, \Lambda)$ ]	4	1.7

<sup>1</sup> All the numerical experiments are defined by  $L = 90$  km,  $\Delta\rho/\rho_0 = 0.5 \times 10^{-3}$ , and  $N = 10^{-2}$  (see Appendix A for a definition of the terms). The velocity scales  $U$  and  $U_b$  are given in the c.g.s. system.

The final residuals of the density, salinity, and temperature equations are depicted in Fig. 6. The highest residual variations are in strong current regions. This suggests

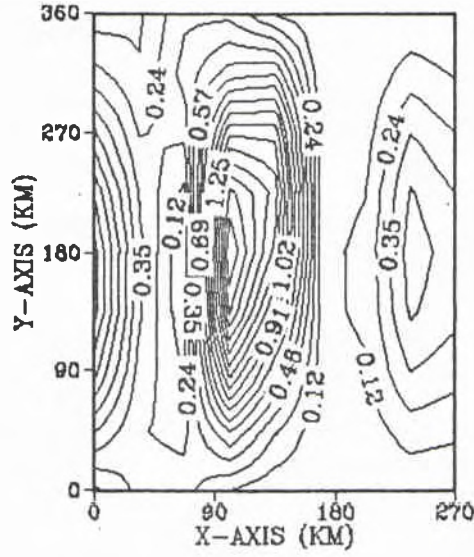


Fig. 5: Amount of energy generated by the solution of the variational principle. [Values are in  $\text{cm}^2 \text{s}^{-2}$ .]

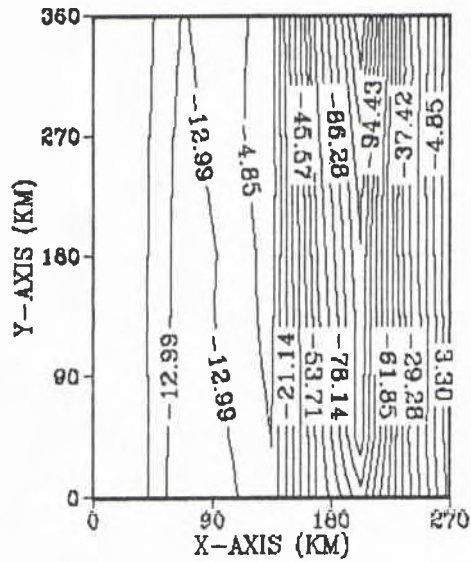


Fig. 6a: Density (scaled by  $10^{12}$ ). [Values are in  $\text{g cm}^{-2} \text{s}^{-1}$ .]

Fig. 6: Vertically averaged residuals of the constituent conservation equations (3.9), computed after application of the variational principle.

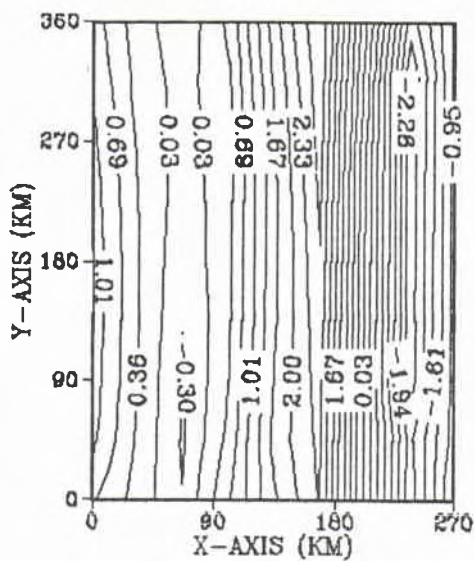


Fig. 6b: Salinity (scaled by  $10^8$ ).  
[Values are in ppt s<sup>-1</sup>.]

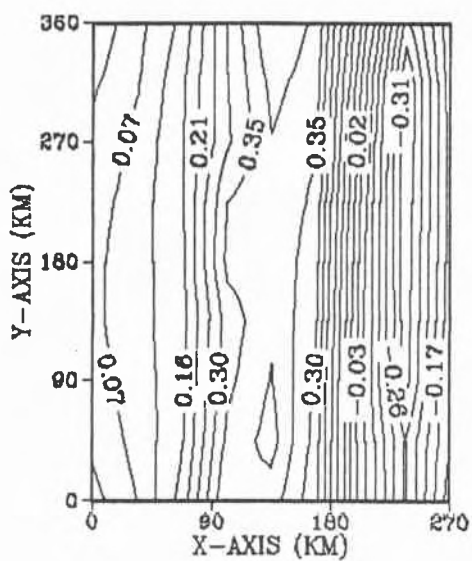


Fig. 6c: Temperature (scaled by  $10^6$ ). [Values are in °C s<sup>-1</sup>.]



that in such regions the flow might be non-linear, obeying a dynamics that has been neglected in our model formulation. Outside the fronts the residuals have smaller gradients, and a range of values that might be representative of mixing processes.

Other numerical experiments have been performed by requesting the minimum misfit for the density alone, and for the salinity and temperature (without density) equations. It has been observed that for the given constituent fields the results are not sensitive to the equations chosen for application of the variational principle. As expected, the residuals of the minimized misfits were smaller, and the residuals of the non-minimized equations were greater, than the corresponding variables of the General Case. However, the differences are irrelevant, and therefore the solutions are not presented here.

(ii) *Effects of noise.* The solution of the General Case was considered noise-free, assuming that density and temperature were exactly measured at the station locations. In this application white noise generated by a random sequence is added to the previous data fields (Fig. 7). The noise has a length scale of the same order of magnitude as the grid spacing, and a maximum amplitude fixed a little above the instrumental accuracy. In practice, observational noise would corrupt the constituent distributions through objective data-analysis smoothing rather than through instrumental errors. However, the theoretical approach of this study did not allow an accurate examination of this aspect of the problem.

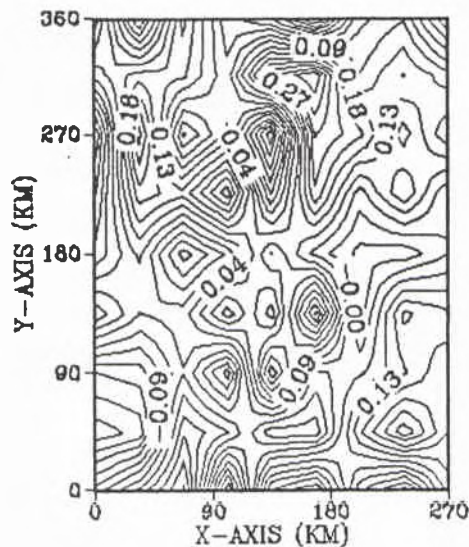
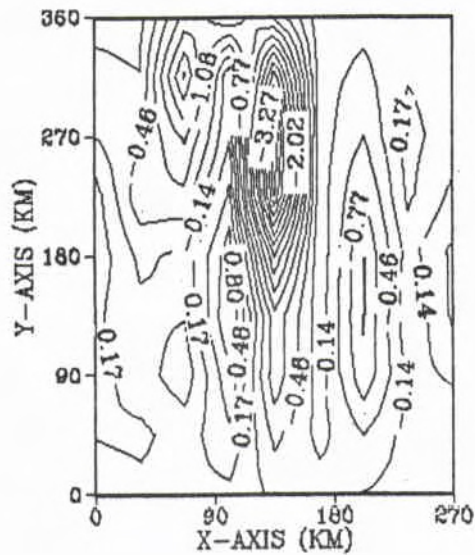
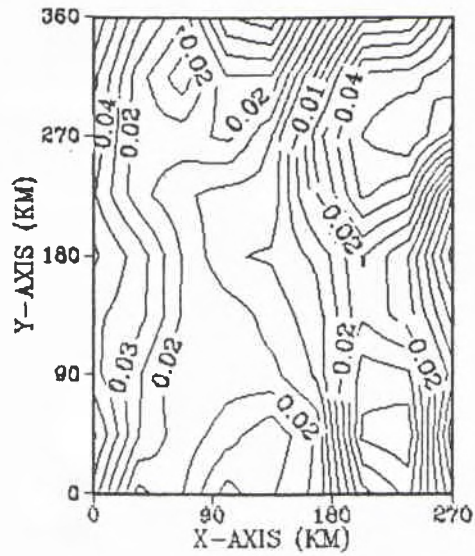


Fig. 7: The  $\sigma$ - $t$  noise distribution at the upper level.

Although the use of noise with a small amplitude does not alter the general behaviour of the solution, it is considered interesting to examine the differences between the two cases. Figs. 8 and 9 illustrate the differences between the energy inputs and the stream functions respectively. It can be seen that various eddies of small intensity are superimposed on the circulation path corresponding to the General Case. They are generated in two ways: locally (through the thermal-wind relationship) and globally (through the solution of the variational problem). However, the contaminated solution shows a length scale greater than the length scale of the noise in the data, which confirms the tendency of the method to smooth the results on a length scale greater than the grid spacing.



*Fig. 8:* Difference between the energies generated by the solutions for smooth and perturbed data fields. [Values are in  $\text{cm}^2 \text{s}^{-2}$ .]



*Fig. 9:* Difference between the stream functions obtained with smooth and perturbed data fields, normalized with respect to the maximum stream function value.

## 5. Summary and conclusion

A variational method for computing the climatic velocity field from the observed density field and other constituent distribution fields in an open, limited region of the ocean is proposed. The method is based on three fundamental assumptions:

- (1) A simplified three-dimensional differential formulation of the equations of motion (i.e. not a multi-layer model) is considered truly representative of the flow dynamics.
- (2) The scale of the motion is *a priori* large in comparison to the station spacing (i.e. it is a global method).
- (3) A valid solution is one which as far as possible is 'consistent with all our observations and our estimates of the accuracy of the model we were using' (Wunsch, 1985).

Starting from these prerequisites, a solution is given in terms of the minimum rms values of the residuals of the constituent conservation equations (3.9) the whole region of interest.

Even though the method does not ask for more than a knowledge of the density field (which implicitly requires direct measurements of temperature and salinity), the incorporation of more constituent equations is advised. Tracers and seawater properties obey a complex dynamics in which both advection and mixing in a turbulent environment are important mechanisms. Although mixing may be quite different from one constituent to another, advection and turbulence depend on the dynamical structure of the flow. Consequently, they are likely to share some of the information contained in each individual constituent. Thus the use of many constituent fields is liable to yield additional information which would improve the quality of the solution. However, 'there is never a guarantee that some new observations will not appear inconsistent with the solution' (Wunsch, 1985).

Comparison between the solution and a diagnostic estimate of the velocity field shows that the method is consistent with the original model formulation. Thus the solution obeys the third of our considerations, and is valid as long as the model equations adequately describe the dynamics of the flow. The model formulation is widely used in the presentation of this inverse problem (Killworth, 1980; Zhdanov and Kamenkovich, 1984; Needler, 1985). The assumptions of geostrophy and hydrostatic balance are traditional, and the thermal wind relation in the interior of the ocean has often been observed. The conservation equations (2.1e) and (3.8) are the aspect most likely to fail. Moreover, errors and noise may corrupt the quality of the hydrographic data. Therefore we consider it correct to act on the

weakest approximations in the model, imposing on the solution the most reliable assumptions. However, certain mechanisms that might be dynamically important are neglected. Perhaps the omission of the influences of the relative vorticity of the flow, and of the boundary layer activities is the major limitation of the model. As has been observed (Fig. 6), the inclusion of nonlinearity may increase the quality of the solution in regions dominated by strong currents, where a high level of turbulence should also be expected. On the other hand, the absence of upper and lower boundary layers in the model precludes consideration of processes (such as atmospheric forcing, dispersion (gain) of heat, evaporation, and dissipation by bottom friction) with effects that might penetrate well inside the inner region of the ocean.

Furthermore, the choice of a global method does not allow a correct application of the model to regions characterized by different, almost-independent regimes (such as coastal open ocean areas). For such a region it could happen that one of the regimes does not obey the model formulations and the solution could try to connect independent features and end up smoothing the differences between the features of the regimes.

With respect to the mathematical framework, the solution of the model is unique, once the norm for the variational principle and the lateral boundary condition for the bottom pressure have been fixed. The calculation of a minimum rms residual is a standard, common choice in geophysical inverse theories. On the other hand, the assumption that the flow across the boundary is zero at the bottom is a more arbitrary decision. However, specification of different boundary conditions for the problem (3.6) (but still consistent with the estimates of Appendix B) does not alter the order of magnitude of the bottom flow.

There is of course, much more to be done before real data may be applied with a reasonable degree of confidence. We have begun a study on the implementation of the method. The use of the variational principle allows us to add coarse, direct velocity measurements, in the formulation of the problem as side conditions imposed on the admissible solution. Finally, it is our belief that the inclusion of the boundary layer dynamics (perhaps specifying upper and lower boundary conditions from estimates and parameterizations of turbulent boundary layers) should be one of the first steps to be taken. Our expectation is to be able to extract information on the mixing coefficients themselves.

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## Appendix A

### Derivation of the model equations

Assume that the ocean is hydrostatic, Boussinesq, incompressible, and inviscid. To describe the motion it is convenient to introduce the departures  $p$  and  $\rho$  from the static equilibrium states of pressure  $p_B$  and density  $\rho_B$  respectively (LeBlond and Mysak, 1978):

$$p_T = p_B(z) + p(x, y, z, t),$$

$$\rho_T = \rho_B(z) + \rho(x, y, z, t),$$

where the subscript T indicates the total fields. Thus the governing equations, written with the same notations used in Sect. 2, are as follows:

$$u_t + uu_x + vu_y + wu_z - fv = -p_x/\rho_0, \quad (\text{A.1a})$$

$$v_t + uv_x + vv_y + wv_z + fu = -p_y/\rho_0, \quad (\text{A.1b})$$

$$0 = -p_z - g\rho, \quad (\text{A.1c})$$

$$u_x + v_y + w_z = 0, \quad (\text{A.1d})$$

$$\rho_t + u\rho_x + v\rho_y + w(\rho + \rho_B)_z = 0, \quad (\text{A.1e})$$

$$f = f_0 + \beta_y. \quad (\text{A.1f})$$

The imposition of no flux across the bottom leads to the boundary condition:

$$w = uh_x + vh_y, \quad \text{at } z = h. \quad (\text{A.1g})$$

The variables are nondimensionalized by assuming geostrophic and hydrostatic balances, i.e.

$$\begin{aligned} (x, y) &= L(\tilde{x}, \tilde{y}), & z &= H\tilde{z}, & (u, v) &= U(\tilde{u}, \tilde{v}), & w &= (UH/L)\tilde{w}, \\ t &= T\tilde{t}, & h &= H\tilde{h}, & (h_x, h_y) &= \lambda(H/L)(\tilde{h}_{\tilde{x}}, \tilde{h}_{\tilde{y}}), \\ p &= \rho_0 f_0 LU\tilde{p}, & \rho &= (\rho_0 f_0 UL/gH)\tilde{\rho}. \end{aligned} \quad (\text{A.2})$$

Introduce the parameters

$$\epsilon = U/f_0L \quad [\text{the Rossby number}] \quad (\text{A.3a})$$

$$N^2 = (-g(\rho_B)_z/\rho_0) \quad [\text{the square of the Brunt-Väisälä frequency}] \quad (\text{A.3b})$$

$$s = (NH/f_0L)^2 \quad [\text{the stratification parameter}] \quad (\text{A.3c})$$

$$\beta_0 = \beta L/f_0 \quad [\text{the planetary vorticity factor}] \quad (\text{A.3d})$$

$$\tau = U/LT \quad (\text{A.3e})$$

and assume the following:

- (i)  $\epsilon \ll 1$  [to explore the departure of motion from geostrophy]
- (ii)  $\tau \ll 1$  [the local time derivative is smaller than the advective time]
- (iii)  $\epsilon \ll \beta_0$  [(relative vorticity)/(planetary vorticity)  $\ll 1$ ]
- (iv)  $\epsilon \ll \Xi = s\beta_0/\epsilon$
- (v)  $\epsilon \ll \Lambda = s\lambda/\epsilon$

Since  $\epsilon \ll 1$ , each variable is expanded in its asymptotic  $\epsilon$ -expansion, with the other parameters directly related to the Rossby number as given above. Dropping the tilde, the 0th-order momentum equations, are

$$-(1 + \beta_0 y)v^0 = -p_x^0, \quad (\text{A.4a})$$

$$(1 + \beta_0 y)u^0 = -p_y^0, \quad (\text{A.4b})$$

From Eqs. (A.4a,b) and the continuity equation (A.1d), the 0th-order potential vorticity equation follows:

$$w_z^0 = \beta_0 v^0. \quad (\text{A.4c})$$

From the boundary condition (A.1g) it also follows that

$$w^0 = \lambda w_1^0 + \beta_0 w_2^0,$$

where  $w_1^0$  is a function that is independent of  $z$ , and  $w_2^0$  is a function such that  $w_2^0(z = h) = 0$ . Thus the mass conservation equation is expressed as follows:

$$u^0 \rho_x^0 + v^0 \rho_y^0 - (\Lambda w_1^0 + \Xi w_2^0) = 0, \quad (\text{A.4d})$$

Equations (A.4) are the nondimensionalized form of the model equations (2.1). We observe that conditions (iv) and (v) are equivalent to assuming that

$$\epsilon \ll (L_D/L_R)^2 \quad [\text{condition (iv)}]$$

$$\epsilon \ll (L_D/L_T)^2 \quad [\text{condition (vi)}]$$

where

$$L_D = (NH/f_0) \quad [\text{the internal Rossby radius of deformation}]$$

$$L_R = (U/\beta_0)^{1/2} \quad [\text{the stationary planetary Rossby wavelength}]$$

$$L_T = (HU/f_0)^{1/2} \quad [\text{the stationary topographic Rossby wavelength} \\ (\text{with } H \text{ representing the topography scale)}]$$



Therefore the ratio of the parameter  $\Lambda$  to the parameter  $\Xi$  expresses the relative importance of topography and Coriolis parameter variations.

Finally, although the conditions (iv) and (v) are necessary for a complete formulation of the problem as given in (2.1), they are not dynamically dependent. The model is still valid even though one of the parameters  $\Lambda$  or  $\beta$  is zero, provided that the dimensionless vertical velocity scale is  $O(1)$  with respect to the parameter  $\epsilon$ .

## Appendix B

An estimate of the bottom velocity relative to the velocity solution of the thermal wind equations

With the same notations used in Sect. 2 and Appendix A, the assumptions of geostrophy and hydrostatic balance imply that the velocity scale  $U$  of (A.2) must be associated with the solution of the thermal wind equations. Furthermore, since the inverse formulation of the problem assumes knowledge of density and its derivatives, it follows from (A.2) that

$$U = \frac{gH}{f_0 L} \frac{\Delta\rho}{\rho_0}, \quad (\text{B.1})$$

where  $\Delta\rho$  is the variation of density on the length scale  $L$ . Thus the velocity vectors  $u_b$  and  $u'$  can be nondimensionalized in the following manner:

$$(u_b, v_b) = U_b(\tilde{u}_b, \tilde{v}_b), \quad (u', v') = U(\tilde{u}', \tilde{v}'). \quad (\text{B.2a})$$

From Eqs. (2.7) and (A.4) the vertical velocity scale follows:

$$w_b = (\lambda + \beta_0) \frac{H}{L} U_b \tilde{w}_b, \quad w' = \beta_0 \frac{HU}{L} \tilde{w}', \quad (\text{B.2b})$$

where the nondimensionalized variables are  $O(1)$  with respect to their  $\epsilon$ -expansion. Thus, the nondimensionalized equation (2.5) is

$$\begin{aligned} \frac{U_b}{f_0 L} (\tilde{u}_b \tilde{\rho}_{\tilde{x}} + \tilde{v}_b \tilde{\rho}_{\tilde{y}}) - s(\lambda + \beta_0) \frac{U_b}{U} \tilde{w}_b &= -\frac{U}{f_0 L} (\tilde{u}' \tilde{\rho}_{\tilde{x}} + \tilde{v}' \tilde{\rho}_{\tilde{y}}) - s\beta_0 \tilde{w}', \\ U_b (\tilde{u}_b \tilde{\rho}_{\tilde{x}} + \tilde{v}_b \tilde{\rho}_{\tilde{y}} - (\Lambda + \Xi) \tilde{w}_b) &= -U (\tilde{\rho}_{\tilde{x}} + \tilde{v}' \tilde{\rho}_{\tilde{y}} - \Xi \tilde{w}'). \end{aligned} \quad (\text{B.3})$$

Let

$$\mu = \tilde{u}' \tilde{\rho}_{\tilde{x}} + \tilde{v}' \tilde{\rho}_{\tilde{y}} = J_{xy}(\tilde{\rho}, \int_{\tilde{h}}^{\tilde{z}} \tilde{\rho}) + \tilde{\rho}(\tilde{z} = \tilde{h}) J_{xy}(\tilde{h}, \tilde{\rho}),$$

where  $J_{xy}$  is the jacobian operator. Thus, we deduce

$$U_b = U \frac{O(\Xi, \mu)}{O(1, \Lambda, \Xi)}, \quad (\text{B.4})$$

where  $O(\star, \star)$  indicates the higher order between the  $O$ 's of each parameter, separately. However, since the horizontal variations of density are usually not strongly dependent on depth above the thermocline, and are almost zero at the bottom, we can consider the parameter  $\mu$  to be generally small, and the velocity scale  $U_b$  to be dependent on topography and stratification.

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