APPROXIMATE METHODS FOR RAY TRACING

by

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INTRODUCTION

The conventional approach to ray tracing is to follow one ray, usually specified by its initial direction, by standard techniques of numerical integration along the ray path, building up a set of values of horizontal displacement, direction, and travel time, as a function of vertical displacement (horizontal stratification of the sound speed profile in the medium is assumed). This process demands computer time and storage. By this means a family of ray plots may be built up [e.g. Fig. 1]

In many applications, however, this forward computation is inconvenient, in that an inverse problem requires solution. Examples are:

a. Given the terminal points of the ray, what is the grazing angle at one point on it (this frequently occurs in experimental determinations of bottom reflectivity).

b. Given the measured travel time from surface to bottom, what is the true slant range (e.g. the use of bottom transponders in some navigational systems).

c. Given the known slant range, what is the true travel time (the converse of (a), also often encountered in bottom studies).

A common feature of these problems is that they do not involve rays having turning points (i.e., the slope is always of the same sign), and basically the theory to be described is restricted to this special, but important, situation. It is possible to extend the treatment to rays having a turning point, but the advantages of the approach are not as marked, and this extension will not be discussed here.

THEORY

Figure 2 shows the geometry of the situation. Horizontal range is denoted by x, vertical depth by z, and the grazing angle at any point on the ray by θ. The terminal points of the ray will be denoted by $(0,0)$ and (X,Z). The sound speed profile (horizontally stratified) is assumed known, the sound speed at depth z being c. The slope of the slant range line θ_0, and the slant range is D.

The basic ray-tracing equations are that along the ray:

$$\tan\theta = dz/dx \quad . \qquad [\text{Eq. } 1]$$

$$\cos\theta/c = p = \text{constant} \quad (\text{Snell's Law}) \quad . \qquad [\text{Eq. } 2]$$

$$dz/dt = c\sin\theta, \text{ where } t = \text{travel time} \quad . \qquad [\text{Eq. } 3]$$

On integrating along the whole path, we obtain

$$X = \int_0^Z \cot\theta \; dz \quad . \qquad [\text{Eq. } 4]$$

$$t = \int_0^Z (\operatorname{cosec}\theta/c)\, dz \quad . \qquad [\text{Eq. } 5]$$

If we use Eq. 2 to express Eq. 4 in terms of c and p, we note that, if given X and Z, Eq. 4 becomes essentially an integral equation for determining p. The method to be described is based on noting that integration is an averaging process, and that this suggests that we are in effect computing some rather complex weighted average of c.

As far as Eq. 5 is concerned, we may note that another average value of c is defined by travel along the straight line path; since by Fermat's principle the true travel time represents a stationary value, the change due to moving to the displaced straight line path can differ only be second order quantities.

We therefore express c in terms of its deviation from the mean value over depth $\bar{c}$, where $\bar{c} = \frac{1}{Z}\int_0^Z c\,dz$, and write $c = \bar{c}[1+\varepsilon(z)]$, where for real profiles

$$\varepsilon(z) \ll 1 \quad . \qquad \text{[Eq. 6]}$$

It is also convenient to replace the Snell's law constant p by an angle $\bar{\theta}$, defined by the equation

$$\cos\bar{\theta}/\bar{c} = p \quad . \qquad \text{[Eq. 7]}$$

Since $\bar{c}$ is a value which actually occurs on the sound speed profile, $\bar{\theta}$ is a real angle for any real ray.

The mean of ε over z is clearly zero, and we may define higher moments by such equations as

$$\overline{\varepsilon^2} = \frac{1}{Z}\int_0^Z \varepsilon^2\,dz \quad . \qquad \text{[Eq. 8]}$$

On making the appropriate substitutions in Eqs. 4 and 5, we obtain

$$X/Z = \cot\theta_0 = \frac{\cot\bar{\theta}}{Z}\int_0^Z (1+\varepsilon)\,[1-\cot^2\bar{\theta}\,(2\varepsilon+\varepsilon^2)]^{-\frac{1}{2}}\,dz \quad . \qquad \text{[Eq. 9]}$$

$$t = (\operatorname{cosec}\bar{\theta}/\bar{c})\int_0^Z (1+\varepsilon)^{-1}\,[1-\cot^2\bar{\theta}\,(2\varepsilon+\varepsilon^2)]^{-\frac{1}{2}}\,dz \quad . \qquad \text{[Eq.10]}$$

Equations 9 and 10 may now be expanded as binomial series in ε, the results being

$$\cot\theta_0 = \cot\bar{\theta}\,[1+{}^3/_2\,\overline{\varepsilon^2}\,\operatorname{cosec}^2\bar{\theta}\,\cot^2\bar{\theta}+O(\overline{\varepsilon^3})] \qquad \text{[Eq.11]}$$

$$\bar{c}t/Z = \operatorname{cosec}\bar{\theta}\,[1+\overline{\varepsilon^2}\,(1-\tfrac{1}{2}\cot^2\bar{\theta}+{}^3/_2\cot^4\bar{\theta})+O(\overline{\varepsilon^3})] \qquad \text{[Eq.12]}$$

the first-order terms vanishing identically.

If we retain only terms to the second order, Eqs. 11 and 12 are very easy to invert or otherwise manipulate, with the following results (noting that $D = Z\operatorname{cosec}\theta_0$) :

$$\cos\bar{\theta} = \cos\theta_0 \, (1 - {}^3/_2 \, \overline{\varepsilon^2} \, \cot^2\theta_0) \qquad \text{[Eq. 13]}$$

$$D = \bar{c}t \, [1 + \tfrac{1}{2} \, \overline{\varepsilon^2} \, \{(\bar{c}^2 \, t^2/z^2) - 3\}] \quad , \qquad \text{[Eq. 14]}$$

and the inverse of Eq. 14

$$\bar{c}t = D[1 - \tfrac{1}{2} \overline{\varepsilon^2}\{(D^2/Z^2) - 3\}] \quad . \qquad \text{[Eq. 15]}$$

These equations clearly give a very simple answer to the problems cited in the introduction. They are easy to compute, and require computer storage for only two environmental parameters, $\bar{c}$ and $\overline{\varepsilon^2}$, both of which are easily computed once for all for any given sound speed profile.

ACCURACY

Equations 13 to 15 are approximations in which terms in $\overline{\varepsilon^3}$ and higher moments have been ignored, and it is obviously necessary to determine the errors introduced (and indeed even to decide if the series is convergent).

This problem has been solved as follows. Considering all possible sound speed profiles for which $\bar{c}$ and $\overline{\varepsilon^2}$ are specified, and for given values of X and Z, for which of these profiles will the values of p or of t given by Eqs. 2, 4 and 5 have extremal values? This is a variational problem, which can be handled by the technique of using Lagrangian multipliers for the equations of condition. The result, for both p and t, is that extremal values will be attained when c(z) is a function of (z) which can take only two discrete values, i.e., when the sound speed profile is that of a two-layered environment.

This, however, is not sufficient to determine a true maximum, since the two-layer profile is specified by only two conditions, but has three degrees of freedom. It is necessary to find a third constraint, and an obvious one is given by the observation that any real profile has bounded values of c, that is, that it has a maximum and a minimum value for sound speed. It can now be shown that the extremal values in this situation will be given when one of the layers is allocated either the greatest or the least value of c, denoted by c_{max} and c_{min} [Fig. 3].

From these extreme profiles it can be shown that the series expansion is absolutely convergent, and that the values of $\cos\bar{\theta}$ and D lie within the bounds given by the following expression:

$$\cos\bar{\theta} = \cos\theta_0 \left[1 - {}^3\!/_2\, \overline{\epsilon^2} \cot^2\theta_0 \, (1+\gamma_1)\right]$$

$$D = \bar{c}t \left[1 + \tfrac{1}{2}\, \overline{\epsilon^2} \,\{(\bar{c}^2 t^2/z^2) - 3\}\, (1+\gamma_2)\right]$$

$$\left(\frac{\overline{\epsilon^2} - b^2}{3b}\right) (5\cot^2\theta_0 - 1) \leq \gamma_1 \leq \frac{a^2 - \overline{\epsilon^2}}{3a} (5\cot^2\theta_0 - 1)$$

$$\left(\frac{\overline{\epsilon^2} - b^2}{b}\right) \cot^2\theta_0 \leq \gamma_2 \leq \left(\frac{a^2 - \overline{\epsilon^2}}{a}\right) \cot^2\theta_0$$

$$c_{max} = \bar{c}(1+a) \qquad c_{min} = \bar{c}(1-b)$$

It is clear from these expression that the error is greatest at the maximum range, and falls off roughly as the fourth power of range.

If we make some simplifying assumptions (basically that gradients are never very large, so that the sound speed profile moves relatively smoothly between its extremes), it can be shown that, to a reasonable degree of accuracy, the error at maximum range is approximately equal to the correction introduced by adding the term in $\overline{\epsilon^2}$ for the Snell's law constant $\cos\bar{\theta}$, and is half the corresponding correction for the slant range determination.

ILLUSTRATIONS

To demonstrate the sort of accuracy that the approximations can give, a comparison has been made between the results of an exact computation, using a digital computer, and the approximations given above, for two profiles (chosen basically to ease the digital computer's task!)

Figure 4 outlines the profiles used. That marked 'typical' has parameters not unlike those found in the real ocean; the 'extreme' profile was designed to have wide limits (a 10% variation in sound speed) and incorporates a marked inversion layer.

Figures 5 and 6 illustrate the results for travel time. The computed slant range errors are shown for

a. the very simple formula $D = \bar{c}t$ and

b. the second-order expression [Eq. 15]. The computer upper and lower bounds are also shown. It will be seen that, with the second order correction terms the error is at most 9 m in 28.9 km for the 'typical profile, and is only 23 m in 19.3 km for the 'extreme' profile.

A similar analysis was carried out to compute the error in initial grazing angle as deduced from the Snell's law parameter $\cos\bar{\theta}$. A summary of the results is given in the following table.

TABLE 1

ERRORS IN GRAZING ANGLE

Profile	Range (km)	True Grazing Angle (deg)	Error in Angle (deg)	Maximum Error Bounds (deg)
'Typical'	15	15	0.01	±0.05
	20	8	0.1	±0.17
	24.5	4	0.27	±0.4
	29.3	0	0.74	±0.87
'Extreme'	10	17.5	0.01	±0.15
	15	7	0.1	±0.8
	16.5	4	0.17	±1.2
	18.7	0	0.29	±1.9

It will be noted that the errors in grazing angle are larger for the 'typical' profile than for the 'extreme'; this is because the horizontal ranges with the 'typical' profile are much greater than for the 'extreme', and the strong range dependence outweighs the smaller variation in the sound speed. Even so, the errors are remarkably small over most of the range, and the accuracy everywhere is probably greater than is warranted by the reliability of the input data.

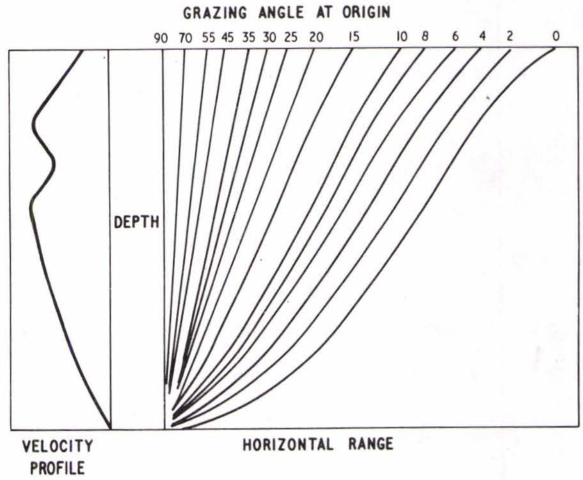
DISCUSSION OF RESULTS

It is apparent that for most purposes the errors introduced by the use of this approach are far smaller than the quality of the input data would justify, and the saving in computer size required is considerable. Furthermore it is clear that, because the sound speed enters only in the form of statistical averages it is easy to assess the precision to which individual measurements should be made. Again, from this analysis, it is evident that the effect of irregularities in the profile will not in general be of great importance; this is a deduction that would be difficult to make by conventional ray-tracing methods.

At first sight the high accuracy of this very simple approximation seems surprising. The following argument gives an explanation for this result. In the integration over z for X and t (Eqs. 4 and 5), the order in which successive increments are added is immaterial, and the profile can be redrawn so that c is a monotonically increasing function of z (this is the same as forming a Lebesgue integral). The approximation then consists, in effect, of replacing this 'regularised' profile by the constant gradient profile of best fit by least squares. The shape of the ray-path will be quite different, but the Snell's law constant and the travel time will be nearly unchanged. This argument also shows immediately why the two-layer profile gives the extreme bounds, since this is the one which is least well fitted by a single straight line.

The method is clearly capable of extension. For example, if in say, a side-scan sonar the launching grazing angle and the travel time are simultaneously recorded it is possible to estimate the height of a projection above the sea-bed (since in effect $\bar{\theta}$ and t are given) by suitable inversion of the equations. Such applications will be reported separately.

FIG. 1



RAY TRACE FAMILY

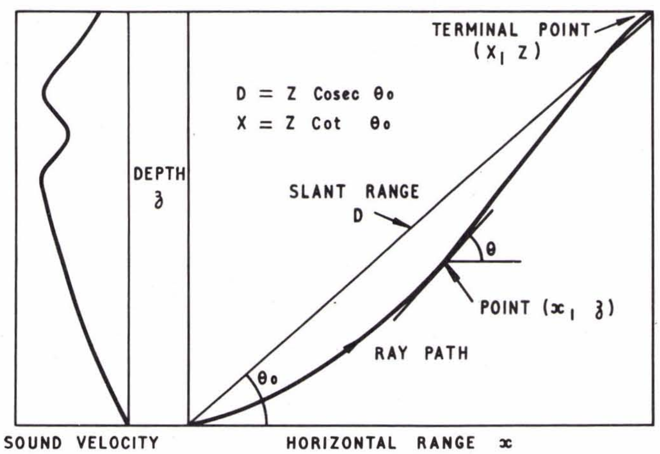
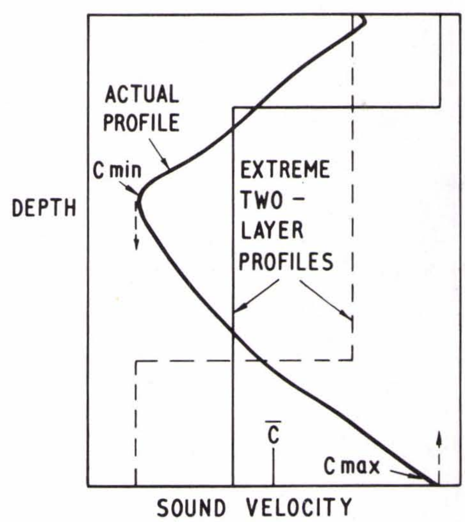


FIG. 2

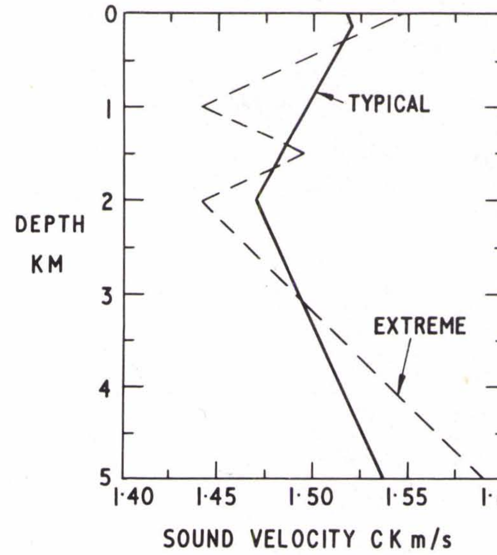
RAY PATH GEOMETRY

FIG. 3



BOUNDING PROFILES

FIG. 4



	EXTREME PROFILE	TYPICAL PROFILE
MEAN $\bar{C}$	1·500	1·500
$\varepsilon^2 = \frac{\overline{C^2}}{\bar{C}^2} - 1$	$7{\cdot}87 \times 10^{-4}$	$1{\cdot}42 \times 10^{-4}$
MAXIMUM VARIATION IN C	10%	4·4%
C MAX	1·590	1·536
C MIN	1·440	1·470

SOUND VELOCITY PROFILES

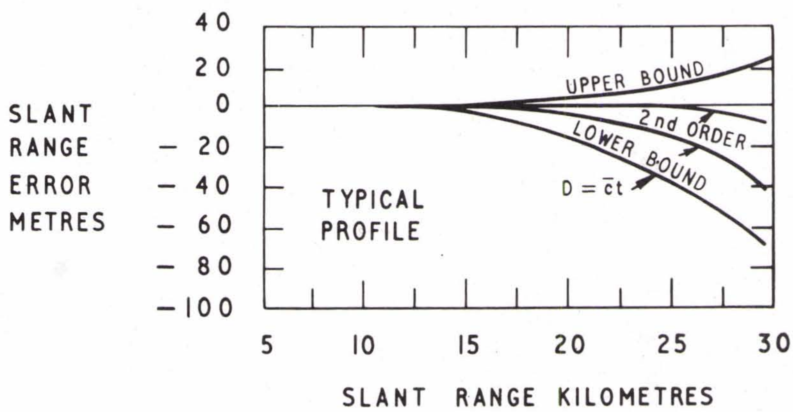
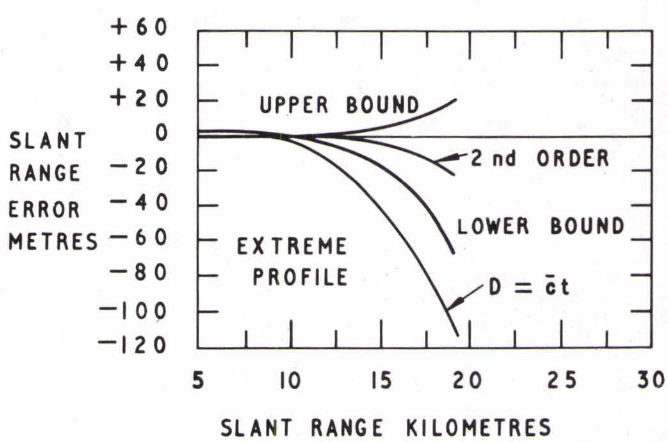


FIG. 5

ERROR IN SLANT RANGE DETERMINATION

FIG. 6



ERROR IN SLANT RANGE DETERMINATION