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A collection of analytical results related to the ASW protection of convoys and carriers

J.G. Pierce

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SACLANT ASW Research Centre
Viale San Bartolomeo 400
19026 San Bartolomeo (SP), Italy

tel: 0187 540 111
telex: 271148 SACENT I

NORTH ATLANTIC TREATY ORGANIZATION

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A collection of analytical results related to the ASW protection of convoys and carriers

J.G. Pierce

Abstract: This paper is a collection of diverse analytical results that are related to several aspects of a common problem: the defence of carriers against torpedo-firing submarines. This work is not in any sense a complete treatment of those issues, but is rather a set of analytical vignettes. Examined are: (1) escort response time to a flaming datum; (2) the probability of detecting the attacker after a flaming datum; (3) ship/submarine exchange ratio as depending on the number of submarine attacks within a convoy; (4) the trade-off between screening and flaming datum prosecution; and (5) carrier survival probability after a submarine has penetrated its screen.

Keywords: aircraft carrier ∘ area ASW ∘ ASW ∘ ASW screen ∘ barriers ∘ carrier ∘ convoy ∘ detection ∘ escort ∘ flaming datum ∘ merchant shipping ∘ protection ∘ screening ∘ survival probability

Contents

1. Introduction

This paper is a collection of diverse analytical results that are related to several aspects of a common problem: the defence of convoys and carriers against torpedo-firing submarines. The work described here is not in any sense a complete treatment of those issues. Rather it is a series of analytical vignettes that help to clarify and simplify the problems, and to justify parameters that can be used in more complex simulations.

One of the overriding considerations in convoy ASW is that a limited number of escorts would be available for protection of the convoy. Consequently, those escorts must be employed in the most efficient manner, in the way that promises the best overall protection to the ships of the convoy. Two modes of employment of ASW escorts are normally considered: (1) in a screen around the forward sectors of the convoy to attempt to detect and destroy submarines before they reach the convoy; and (2) embedded in the body of the convoy to respond urgently to submarine attacks on merchant ships (flaming datums) and thereby forestall further attacks. The relative merit of these two modes of employment is a major analytical issue. The next four sections of this paper contain supporting results that can be used in addressing that issue.

Section 2 discusses the statistics of the distance between a randomly located flaming datum and an escort embedded in the convoy. It is shown that the root mean square travel time for response to a flaming datum is proportional to $\sqrt{n}$, where n is the number of escorts within the convoy.

Section 3 is a simple tactical model of the prosecution of a flaming datum. The probability of successfully detecting the submarine after a merchant ship has been torpedoed is expressed in terms of several more fundamental tactical parameters. The sensitivity of the detection probability to each of these parameters is tested.

Section 4 discusses the ship/submarine exchange ratio as a function of submarine attack policy—i.e. the number of attacks attempted after penetrating a convoy.

Section 5 develops some of the considerations that are important for evaluating the trade-off between escorts in the forward screen and escorts within the convoy.

Finally, section 6 of the paper addresses a somewhat different issue. A carrier is assumed to be operating in a confined area. Submarines may enter the area by penetrating a defensive barrier. Once inside, they are subject to attack by area ASW forces and by ASW screens if they encounter the carrier. If the submarines survive long enough they may attack and sink the carrier. The analytical issue is to determine the long-term probability of survival of the carrier.

The method developed in Sect. 6 provides a graphical way of determining that survival probability. The relative influence of barriers, area ASW, and ASW screens can then be calculated.

2. Geometrical aspects of flaming-datum prosecution

In connection with the prosecution of flaming datums, it is important to know the statistics—at least the mean and the variance—of the distance between the position of an ASW unit and the position at which the flaming datum occurs. When these are known, the mean travel time to the flaming datum can be calculated trivially.

Let the area covered by a convoy be A. If there are n ASW escorts, then each escort is responsible for a subarea of size $a = A/n$. If the escort is in the geometric center of its area and a submarine attack on a merchant vessel occurs at random, we wish to determine the average distance that the escort must travel to reach the flaming datum.

We will consider various geometries. The circular geometry yields the simplest calculation, but is unrealistic in practice. The more plausible rectangular geometry leads to rather cumbersome mathematics.

For the *circular case*,

$$a = \pi R^2 \tag{1}$$

and we assume that the flaming datum is a random event, uniformly, distributed within the circle of radius R. Then the distance d to the flaming datum is simply

$$d = r, \qquad 0 \leq r \leq R. \tag{2}$$

Thus

$$\begin{aligned} \bar{d} &= \frac{1}{\pi R^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^R r\, r\, \mathrm{d}r \\ &= \frac{2}{3} R = \frac{2}{3} \sqrt{\frac{a}{\pi}} = 0.376\sqrt{a}\,, \end{aligned} \tag{3}$$

$$\begin{aligned} \overline{d^2} &= \frac{1}{\pi R^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^R r^2\, r\, \mathrm{d}r \\ &= \frac{1}{2} R^2. \end{aligned} \tag{4}$$

Consequently

$$\sigma_{dc}^2 = \overline{d^2} - \overline{d}^2 = \frac{1}{2}R^2 - \frac{4}{9}R^2 = \frac{1}{18}R^2, \tag{5}$$

so

$$\sigma_{dc} = 0.133\sqrt{a}. \tag{6}$$

We keep these simple results for comparison with the more complicated rectangular case.

For the *rectangular case*, we introduce an additional parameter that will allow us to treat two special cases, one in which the escort is initially inside the rectangle and one in which the escort is outside the rectangle.The latter case would apply when the escort is in a screen ahead of the convoy.

Figure 1 defines the variables.

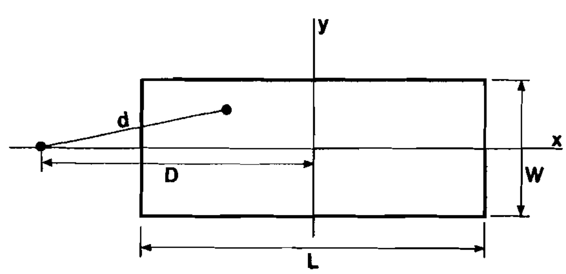

Fig. 1: A diagram of the rectangular case for area L by W. D represents the distance of the escort from the center of the area, and d from the flaming datum.

Here, $d = \sqrt{(x + D)^2 + y^2}$, where x and y are uniformly distributed in the rectangle.

Then

$$\overline{d} = \frac{1}{L\omega}\int_{-\omega/2}^{\omega/2} \mathrm{d}y \int_{-L/2}^{L/2} \mathrm{d}x \sqrt{(x + D)^2 + y^2}. \tag{7}$$

Let $z = x + D$, and perform the y-integration first:

$$\overline{d} = \frac{1}{L\omega}\int_{D-L/2}^{D+L/2} \mathrm{d}z \frac{1}{2}\left\{\omega\sqrt{z^2 + \omega^2/4} + z^2 \ln\left[\frac{\sqrt{z^2 + \omega^2/4} + \omega/2}{\sqrt{z^2 + \omega^2/4} - \omega/2}\right]\right\}. \tag{8}$$

Next, let $\frac{1}{2}\omega t = z$ and $t = 2z/\omega$:

$$\overline{d} = \frac{1}{2L\omega}\int_{(2D-L)/\omega}^{(2D+L)/\omega} \mathrm{d}t \frac{\omega^3}{8}\left\{2\sqrt{t^2+1} + t^2 \ln\left[\frac{\sqrt{t^2+1}+1}{\sqrt{t^2+1}-1}\right]\right\}$$

$$= \frac{\omega^2}{16L}(I_1 + I_2), \tag{9}$$

where

$$I_1 = 2\int_{(2D-L)/\omega}^{(2D+L)/\omega} \mathrm{d}t\sqrt{t^2+1}, \tag{10}$$

$$I_2 = \int_{(2D-L)/\omega}^{(2D+L)/\omega} \mathrm{d}t \ln\left[\frac{\sqrt{t^2+1}+1}{\sqrt{t^2+1}-1}\right]. \tag{11}$$

I_1 can be found immediately in standard tables:

$$I_1 = \frac{2D+L}{\omega^2}\sqrt{(2D+L)^2+\omega^2}$$

$$- \frac{2D-L}{\omega^2}\sqrt{(2D-L)^2+\omega^2}$$

$$+ \ln\left[\frac{2D+L+\sqrt{(2D+L)^2+\omega^2}}{2D-L+\sqrt{(2D-L)^2+\omega^2}}\right]. \tag{12}$$

To calculate I_2 we must integrate by parts. Let

$$u = \ln\left[\frac{\sqrt{t^2+1}+1}{\sqrt{t^2+1}-1}\right], \tag{13}$$

$$\mathrm{d}V = t^2\,\mathrm{d}t. \tag{14}$$

Then

$$V = \frac{1}{3}t^3, \tag{15}$$

and

$$\mathrm{d}u = \frac{t}{\sqrt{t^2+1}}\left[\frac{1}{\sqrt{t^2+1}+1} - \frac{1}{\sqrt{t^2+1}-1}\right]\mathrm{d}t$$

$$= \frac{-2\,\mathrm{d}t}{t\sqrt{t^2+1}}. \tag{16}$$

Consequently

$$
\begin{aligned}
I_2 &= \int U\,\mathrm{d}V = UV| - \int V\,\mathrm{d}U \\
&= UV| + \frac{2}{3}\int \frac{t^2\,\mathrm{d}t}{t^2+1} \\
&= UV| + \frac{2}{3}\left\{\frac{t}{2}\sqrt{t^2+1} - \frac{1}{2}\ln(\sqrt{t^2+1}+t)\right\} \\
&= \frac{1}{3}\left(\frac{2D+L}{\omega}\right)^3 \ln\left[\frac{\sqrt{(2D+L)^2+\omega^2}+\omega}{\sqrt{(2D+L)^2+\omega^2}-\omega}\right] \\
&\quad - \frac{1}{3}\left(\frac{2D-L}{\omega}\right)^3 \ln\left[\frac{\sqrt{(2D-L)^2+\omega^2}+\omega}{\sqrt{(2D-L)^2+\omega^2}-\omega}\right] \\
&\quad + \frac{1}{3}\left(\frac{2D-L}{\omega^2}\right)\sqrt{(2D+L)^2+\omega^2} \\
&\quad - \frac{1}{3}\left(\frac{(2D-L)}{\omega^2}\right)\sqrt{(2D-L)^2+\omega^2} \\
&\quad - \frac{1}{3}\ln\left[\frac{\sqrt{(2D+L)^2+\omega^2}+2D+L}{\sqrt{(2D+L)^2+\omega^2}+2D-L}\right]. \qquad (17)
\end{aligned}
$$

Then, combining I_1 and I_2 we get

$$
\begin{aligned}
I_1 + I_2 &= \frac{4}{3}\left[\frac{2D+L}{\omega^2}\sqrt{(2D+L)^2+\omega^2} - \frac{2D-L}{\omega^2}\sqrt{(2D-L)^2+\omega^2}\right] \\
&\quad + \frac{2}{3}\ln\left[\frac{\sqrt{(2D+L)^2+\omega^2}+2D+L}{\sqrt{(2D+L)^2+\omega^2}+2D-L}\right] \\
&\quad + \frac{1}{3}\left\{\left(\frac{2D+L}{\omega}\right)^3 \ln\left[\frac{\sqrt{(2D+L)^2+\omega^2}+\omega}{\sqrt{(2D+L)^2+\omega^2}-\omega}\right]\right. \\
&\quad \left. - \left(\frac{2D-L}{\omega}\right)^3 \ln\left[\frac{\sqrt{(2D-L)^2+\omega^2}+\omega}{\sqrt{(2D-L)^2+\omega^2}-\omega}\right]\right\}. \qquad (18)
\end{aligned}
$$

Finally

$$\bar{d} = \frac{1}{12}\left[\frac{2D+L}{L}\sqrt{(2D+L)^2+\omega^2} - \frac{2D-L}{L}\sqrt{(2D-L)^2+\omega^2}\right]$$
$$+ \frac{1}{24}\frac{\omega^2}{L}\ln\left[\frac{\sqrt{(2D+L)^2+\omega^2}+2D+L}{\sqrt{(2D+L)^2+\omega^2}+2D-L}\right]$$
$$+ \frac{1}{48}\left\{\frac{(2D+L)^3}{\omega L}\ln\left[\frac{\sqrt{(2D+L)^2+\omega^2}+\omega}{\sqrt{(2D+L)^2+\omega^2}-\omega}\right]\right.$$
$$\left. - \frac{(2D-L)^3}{\omega L}\ln\left[\frac{\sqrt{(2D-L)^2+\omega^2}+\omega}{\sqrt{(2D-L)^2+\omega^2}-\omega}\right]\right\}. \qquad (19)$$

From this general result, we are particularly interested in two special cases.

Case A. $D = 0$. This corresponds to the situation in which the escort is in the middle of the rectangle.

Case B. $D \gg W$. This corresponds to the case in which the escort is ahead of the convoy, and has responsibility for responding in a strip behind its patrol sector.

For *case A*,

$$\bar{d} = \frac{1}{6}\sqrt{L^2+\omega^2} + \frac{1}{24}\frac{\omega^2}{L}\ln\left[\frac{\sqrt{L^2+\omega^2}+L}{\sqrt{L^2+\omega^2}-L}\right]$$
$$+ \frac{1}{24}\frac{L^2}{\omega}\ln\left[\frac{\sqrt{L^2+\omega^2}+\omega}{\sqrt{L^2+\omega^2}-\omega}\right]. \qquad (20)$$

In the case of a square area, $\omega = L$ and

$$\bar{d} = L\left(\frac{\sqrt{2}}{6} + \frac{1}{12}\ln\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]\right)$$
$$= 0.383\sqrt{a}. \qquad (21)$$

Comparing this with the results for the circle, we see that the difference is less than 2%.

For *case B*,

$$\overline{d} = D + \theta(W^2/D^2). \tag{22}$$

Thus to first order, the mean distance that the escort must travel is just the distance from its patrol position to the midpoint of the strip in the convoy behind it.

Next we calculate the variance:

$$\sigma_{dR}^2 = \overline{d^2} - \overline{d}^2. \tag{23}$$

Unlike $\overline{d}$, the calculation of $\overline{d^2}$ is trivial:

$$\begin{aligned}\overline{d^2} &= \frac{1}{L\omega}\int_{\omega/2}^{\omega/2} \mathrm{d}y \int_{L/2}^{L/2} \mathrm{d}x\, \left((x+D)^2 + y^2\right) \\ &= D^2 + 1/12(L^2 + W^2).\end{aligned} \tag{24}$$

This can be used to calculate the special cases.

For *case A* with $W = L$,

$$\overline{d^2} = L^2/6 \tag{25}$$

and

$$\sigma_{dR}^2 = L^2\left(1/6 - (0.383)^2\right), \tag{26}$$

so

$$\sigma_{dR} = 0.141\sqrt{a}, \tag{27}$$

as compared with

$$\sigma_{dc} = 0.133\sqrt{a} \tag{28}$$

for the circle, a difference of 6%.

For *case B*,

$$\sigma_{dR} \cong L/\sqrt{12} = 0.289L. \tag{29}$$

Finally, since the transit time to the position of the flaming datum is

$$\overline{T}_{TR} = \overline{d}/s, \tag{30}$$

we have

$$\overline{T}_{TR} = \frac{k}{s}\frac{\sqrt{A}}{\sqrt{n}}, \tag{31}$$

where k is the appropriate numerical constant.

3. A simple model of flaming datum prosecution

To analyse the prosecution of a flaming datum, we assume that a submarine succeeds in penetrating the defenses of a convoy and enters the interior of the convoy. The convoy dimensions and size are $B \times L = A$.

The submarine fires torpedoes from a range R_T, which strike a target at $t = 0$. The target sends an emergency message which is received and acted upon by nearby ASW escorts after a communication delay T_C. If the submarine begins escape manoeuvers immediately upon firing the torpedoes, the total time from torpedo firing until ASW forces reach the scene is

$$T_L = T_{TD} + T_C + T_{TR}, \tag{32}$$

where T_{TD} is the torpedo run time, T_C is the communication and response time, and T_{TR} is the transit time of the ASW units. Thus, the area of uncertainty of the submarine's position at T_L is $a(T_L) = \pi(R_T + VT_L)^2$, where V is submarine escape speed. This area of uncertainty continues to grow as the ASW forces search the area.

Let $P(t)$ be the probability of detecting the submarine at time t, and let $Q(t) = 1 - P(t)$. Then

$$Q(t + \Delta t) = Q(t)(1 - \lambda(t)\Delta t), \tag{33}$$

where $\lambda(t)$ is the probability of detection in incremental time Δt. This can be integrated to give

$$Q(t_2) = Q(t_1)\exp\left[-\int_{t_1}^{t_2} \lambda(t)\,\mathrm{d}t\right]. \tag{34}$$

The detection probability λ can be expressed as the ratio of the area swept in Δt to the total area of uncertainty, or

$$\lambda(t) = SW/a(t), \tag{35}$$

where S is the speed of the searcher and W his sweep width. Consequently

$$\begin{aligned}\int_{t_1}^{t_2} \lambda(t)\,\mathrm{d}t &= \frac{SW}{\pi}\int_{t_1}^{t_2} \frac{\mathrm{d}t}{(R_T + Vt)^2} \\ &= \frac{1}{\pi}\frac{S}{V}\frac{W}{R_T}\int_{R_T t_2/V}^{R_T t_1/V} \frac{\mathrm{d}z}{(1+z)^2}.\end{aligned} \tag{36}$$

If the search begins when the ASW forces arrive, then $t_1 = T_L$ and $t_2 = t$, so

$$Q(t) = Q(T_L)\exp\left[\frac{SW}{\pi V R_T}\left\{\frac{1}{1 + Vt/R_T} - \frac{1}{1 + VT_L/R_T}\right\}\right]. \tag{37}$$

Since $Q(T_L) = 1$,

$$P(t) = 1 - \exp\left[\frac{SW}{\pi V R_T}\left\{\frac{1}{1 + Vt/R_T} - \frac{1}{1 + VT_L/R_T}\right\}\right]. \tag{38}$$

Finally, we look for the probability that the evading submarine is detected, i.e. for $P(\infty)$. This is

$$P(\infty) = 1 - \exp\left[-\left(\frac{SW}{\pi V R_T}\right)\left(\frac{1}{VT_L/R_T}\right)\right]. \tag{39}$$

In this expression, the parameters combine in two dimensionless groups so only two parameters are needed to describe the probability

$$P_\infty(\alpha, \beta) = 1 - \exp\left[-\frac{\alpha}{1 + \beta}\right], \tag{40}$$

where

$$\alpha = SW/VR_T\pi, \tag{41}$$

$$\beta = VT_L/R_T. \tag{42}$$

The ranges $0.1 \leq \alpha \leq 1$ and $1 \leq \beta \leq 10$ will cover most situations of practical interest. Values of P_∞ for this range of parameters are shown in Table 1.

TABLE 1
Sensitivity of $P_\infty(\alpha, \beta)$ to variations in parameters α and β

β	$P_\infty(\alpha,\beta)$				
	α =0.2	0.4	0.6	0.8	1.0
2	0.064	0.125	0.181	0.234	0.283
4	0.039	0.077	0.113	0.148	0.187
6	0.028	0.056	0.082	0.108	0.133
8	0.022	0.043	0.064	0.085	0.105
10	0.018	0.036	0.053	0.070	0.087

Within the confines of this table, we can also investigate the sensitivity of P_∞ to variations in the individual parameters. We choose the following nominal values:

$$V = 20\,\text{kn}, \qquad S = 20\,\text{kn},$$

$$R_T = 3\,\text{n.mi}, \qquad W = 4\,\text{n.mi},$$

$$T_L = 0.5\,\text{h}.$$

Then the nominal value of P_∞ is

$$\hat{P}_\infty = 1 - \exp\left[-\left(\frac{4\times 20}{\pi\times 20\times 3}\right)\left(\frac{1}{1+0.5\times 20/3}\right)\right]$$
$$= 0.09. \tag{43}$$

Then, in terms of the reduced variables

$$v = 1/20V, \qquad s = 1/20S,$$
$$r_T = 1/3R_T, \qquad w = 1/4W,$$
$$t_L = 2T_L,$$

we have

$$P_\infty(v) = 1 - \exp\left[-\left(\frac{0.424}{v+3.33v^2}\right)\right], \tag{44}$$

$$P_\infty(s) = 1 - \exp\left[-(0.098s)\right], \tag{45}$$

$$P_\infty(w) = 1 - \exp\left[-(0.098w)\right], \tag{46}$$

$$P_\infty(r_T) = 1 - \exp\left[-\left(\frac{0.424}{r_T+3.33}\right)\right], \tag{47}$$

$$P_\infty(t_L) = 1 - \exp\left[-\left(\frac{0.424}{1+3.33t_L}\right)\right]. \tag{48}$$

These equations are plotted in Figs. 2 to 6, showing the effect of changing one variable at a time while the others are held at their nominal values. The results suggest the following:

(1) Detection probability is highly sensitive to submarine escape speed, especially at the lower vales ($\leq$ 15 kn). This parameter is not, of course, under the control of the search forces. However, if the submarine does use a low escape speed, for reasons of low battery state, casualties, or misguided attempts to hide rather than run, the probability of detection is much improved. For a nuclear submarine or a modern diesel with a high battery state, escape can be made above 15 kn, and the detection probability is correspondingly low. Above 20 kn, marginal increases in escape speed make little difference in the already low detection probability.

(2) Detection probability is quite sensitive to the sweep width of the searcher. This parameter depends on the type and quality of the availability sensors, as well as on environmental conditions. Tactical commanders should try to emploly the sensors with the largest possible sweep width in flaming datum search. For example, a sonobuoy field would normally be considerably superior to active sonar in searching for a nuclear target.

(3) Search speed has a strong influence on detection probability in principle, since the functional relationship is the same as that for sweep width. In practice, however, the range of attainable search speeds is so narrow that large gains in detection probability are not practical.

(4) The range at which the submarine fires torpedoes has very little bearing on the subsequent detection probability. Changes in firing range by a factor of 6 only change the detection probability by about 3%.

(5) Detection probability is highly sensitive to time late if the time late is under 15 min. For much larger values of time late, the marginal change in detection probability is insignificant.

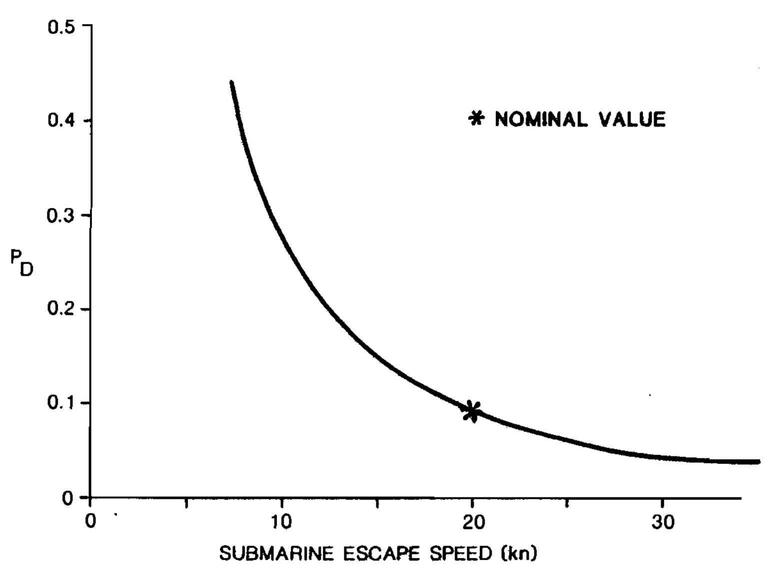

Fig. 2: Probability that the evading submarine is detected as a function of submarine escape speed.

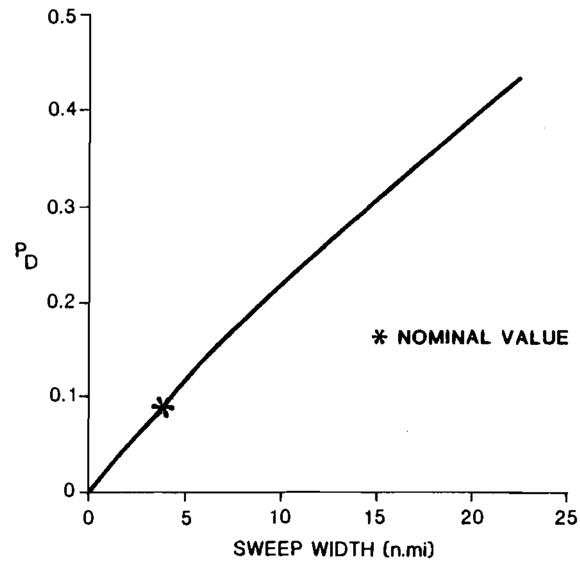

Fig. 3: Probability that the evading submarine is detected as a function of sweep width.

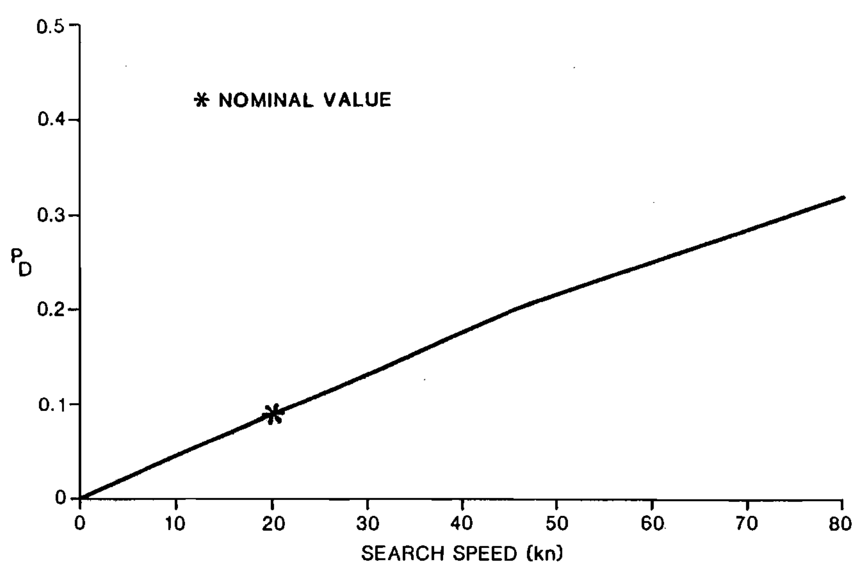

Fig. 4: Probability that the evading submarine is detected as a function of search speed.

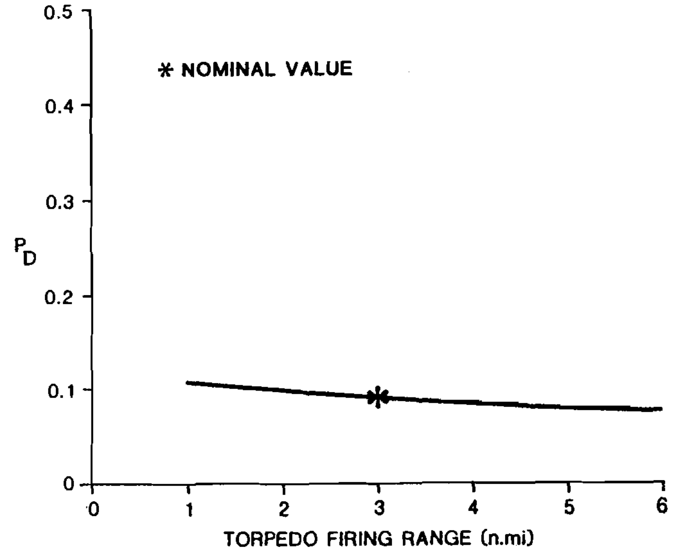

Fig. 5: Probability that the evading submarine is detected as a function of torpedo firing range.

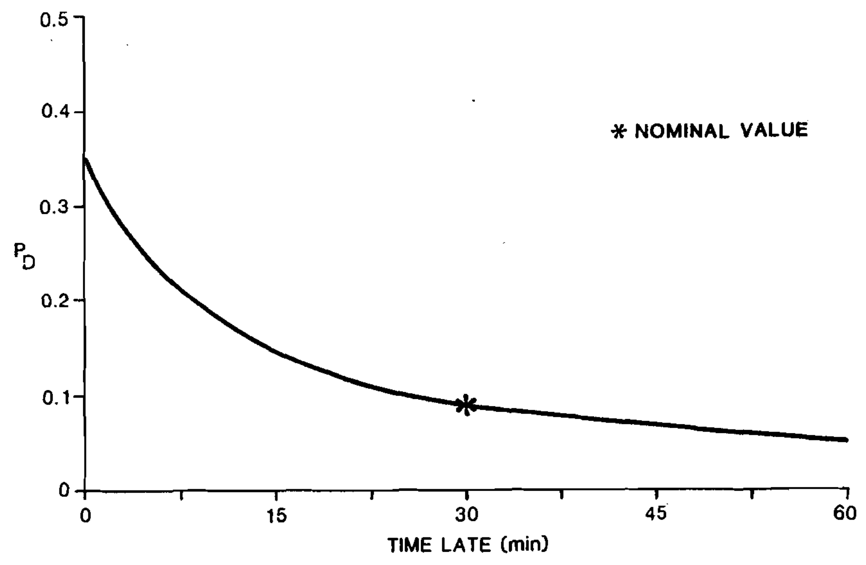

Fig. 6: Probability that the evading submarine is detected as a function of time late.

4. The value of multiple attacks from the submarine's perspective

In the previous section it was assumed that the submarine entered the convoy freely and that the ASW action began with a flaming datum. In reality, the submarine does not have free entry to the convoy. It must pass through one or more protective screens and undergo some risk of being sunk before it ever reaches the convoy. This initial risk changes the utility to the submarine of multiple attacks once it is inside the convoy. Having survived the outer barriers, the submarine has greater incentive to make multiple attacks despite the added risk that those entail. This section illustrates this point with a simple calculation.

An approximate expression for the warship/submarine exchange ratio in a convoy attack can be derived as follows.

Let μ = probability that submarine is detected and killed by the screen before entering the convoy, γ = probability that a submarine attack on a ship is successful, and ν = probability that the submarine is detected and killed as the result of a flaming datum. We assume that there is no flaming datum if the submarine's torpedoes miss their target, so we have the following probabilities for each submarine attack:

ship sunk: γ	sub sunk: $\gamma\nu$
ship not sunk: $1-\gamma$	sub not sunk: $1-\gamma\nu$

The exchange ratio per attack within the convoy is thus

$$\frac{P(\text{ship sunk})}{P(\text{sub sunk})} = \frac{\gamma}{\gamma\nu} = \frac{1}{\nu}. \tag{49}$$

Assuming that the attacks are independent, the probability that the submarine is sunk after carrying out N attacks is

$$P(\text{sub sunk}) = 1 - (1-\gamma\nu)^N. \tag{50}$$

N will be limited by torpedo load and firing doctrine. With a load of 20 torpedoes, N is 5 if they are fired 4 at a time.

Because the exchange per attack is fixed, the expected number of ships sunk, given penetration of the convoy, is

$$\begin{aligned} E(\text{ship sunk}) &= \frac{1}{\nu}P(\text{sub sunk}) \\ &= \frac{1}{\nu}[1-(1-\gamma\nu)^N]. \end{aligned} \tag{51}$$

This intuitive result can be confirmed by a careful probability analysis of the possible outcomes of N attacks. The overall exchange ratio is then

$$R = \frac{0\mu + (1-\mu)\nu^{-1}(1-(1-\gamma\nu)^N)}{\mu + (1-\mu)(1-(1-\gamma\nu)^N)}$$

$$= \frac{\nu^{-1}(1-(1-\gamma\nu)^N)}{\mu(1-\mu)^{-1} + (1-(1-\gamma\nu)^N)}. \tag{52}$$

As a simple example, for $\mu = 0.2$, $\nu = 0.15$ and $\gamma = 0.6$ we find

TABLE 2
Some sample numerical values from Eq. (52)

N	P(sub sunk)	E(ship sunk)	Ratio
1	0.28	0.53	1.91
2	0.35	1.01	2.88
3	0.42	1.44	3.46
4	0.48	1.82	3.84
5	0.53	2.18	4.14

Equation (52) shows that for $\mu = 0$, R reduces to $1/\nu$, independent of N, which is 6.67 in the example. The consequence of a non-zero μ is to reduce the exchange ratio very drastically if only one attack is made. The effect of repeated attacks by the submarine is to increase the exchange ratio in its favor, approaching

$$R = \frac{1-\mu}{\nu} < \frac{1}{\nu} \tag{53}$$

as N gets large.

5. Trade-offs between ASW screening and prosecution of flaming datums

An important problem in the protection of convoys is to understand the trade-offs between screening and prosecution of flaming datums. There are several important considerations in that problem, only one of which is treated in this section.

Here we consider only the problem of detecting an evading submarine. Issues related to the prosecution of the target are deferred, as are those related to the relative value of detecting a target in the screen rather than after it has torpedoed a ship in the convoy.

The simple model is as follows. N escorts are available for protection of a convoy. Of these, n are embedded within the convoy for the purpose of responding to flaming datums. The remaining $N - n$ are assigned to the screen to attempt to detect enemy submarines before they reach attack range. We will be interested primarily in the overall probability of detecting the submarine as a function of the allocation of the escorts among the screen and the interior of the convoy.

Let P_{DS} be the probability of detection in the screen and P_{DF} the probability of detection at the flaming datum. Then, because these refer to independent events, the overall probability of detection is

$$P_D = 1 - (1 - P_{DS})(1 - P_{DF}). \tag{54}$$

In Sect. 3 it was shown that

$$P_{DF} = 1 - e^{-\Phi_F}, \tag{55}$$

where

$$\Phi_F = \frac{SW/VR_T\pi}{1 + VT_L/R_T}. \tag{56}$$

(See Sect. 3 for definitions.)

In many practical cases it is also possible to write P_{DS} in a similar form:

$$P_{DS} = 1 - e^{-\Phi_S}. \tag{57}$$

When this is the case,

$$\begin{aligned} P_D &= 1 - (1 - (1 - e^{-\Phi_S}))(1 - (1 - e^{-\Phi_F})) \\ &= 1 - e^{-(\Phi_S + \Phi_F)} = 1 - e^{-\Phi}. \end{aligned} \tag{58}$$

Both Φ_S and Φ_F are functions of n. Consequently

$$P_D(n) = 1 - e^{-\Phi(n)}, \tag{59}$$

and, if we allow ourselves the liberty of treating n as a continuous variable,

$$\frac{\mathrm{d}}{\mathrm{d}n} P_D(n) = e^{-\Phi(n)} \frac{\mathrm{d}\Phi}{\mathrm{d}n} = 0 \tag{60}$$

implies that the optimum allocation of escorts can be found by setting

$$\Phi'(n) = 0. \tag{61}$$

In practice, the nearest integer n is nearly optimum.

For a low-density screen trying to cover a front of length F, with $N - n$ ships, each having a sweep width W, the detection probability is

$$P_{DS} \simeq (N - n)W/F. \tag{62}$$

This can be viewed as an approximation to the high-density expression

$$\begin{aligned} P_{DS} &= 1 - (1 - W/F)^{N-n} \\ &= 1 - e^{(N-n)} \ln(1 - W/F). \end{aligned} \tag{63}$$

Thus

$$\Phi_S(n) = N|\ln(1 - W/F)| - n|\ln(1 - W/F)|,$$

so

$$\Phi_S(n) = a - bn,$$

where a and b are positive constants.

In Sect. 3, Eq. (32), and Sect. 2, Eq. (31), we have seen that

$$T_L = T_{TD} + T_C + T_{TR}, \tag{64}$$

$$\overline{T}_{TR} = \frac{\overline{d}}{S} = k\frac{\sqrt{a}}{S} = \frac{k\sqrt{A}}{S\sqrt{n}}. \tag{65}$$

In this expression, $\overline{d}$ is the average distance that an escort must travel to reach the flaming datum, A is the area of the whole convoy and $a = A/n$ is the area assigned to each escort, and k is the numerical constant

$$k \simeq 0.38\,. \tag{66}$$

From this we see that

$$\Phi_F = \frac{SW/VR_T\pi}{1 + V/R_T(T_{TD} + T_C) + Vk/R_T(\sqrt{A}/S1/\sqrt{n})}. \qquad (67)$$

This may then be written

$$\Phi_F = \frac{c\sqrt{n}}{d + \sqrt{n}}, \qquad (68)$$

where

$$c = \frac{SW/\pi VR_T}{1 + (V/R_T)(T_{TD} + T_C)}, \qquad (69)$$

$$d = \frac{VK\sqrt{A}/R_TS}{1 + (V/R_T)(T_{TD} + T_C}. \qquad (70)$$

Both c and d are positive constants.

Then

$$\Phi = a - bn + \frac{c\sqrt{n}}{d + \sqrt{n}}. \qquad (71)$$

With

$$\rho = \sqrt{n} \qquad (72)$$

we have

$$\Phi = a - b\rho^2 + \frac{c\rho}{d + \rho}, \qquad (73)$$

$$\frac{d\Phi}{d\rho} = -2b\rho + \frac{dc}{(d + \rho)^2} = 0, \qquad (74)$$

or

$$\rho(\rho + d)^2 = \frac{dc}{2b} \equiv \lambda. \qquad (75)$$

Making the further substitutions

$$X = \rho/d, \qquad \mu = \lambda/d^3, \qquad (76)$$

we obtain

$$X(X + 1)^2 = \mu. \qquad (77)$$

This has one real solution and two complex solutions. The real solution can be written

$$
\begin{aligned}
X = &\left(1/27 + 1/2\mu + \sqrt{1/27\mu + 1/4\mu^2}\right)^{1/3} \\
&+ \left(1/27 + 1/2\mu - \sqrt{1/27\mu + 1/4\mu^2}\right)^{1/3} \\
&- 2/3.
\end{aligned}
\tag{78}
$$

This then gives us

$$
X \sim \begin{cases} 0.385\mu^{1/2}, & \text{for } \mu \to 0; \\ \mu^{1/3} - 0.667, & \text{for } \mu \to \infty. \end{cases}
$$

Figure 7 shows $X(\mu)$ over the full range of μ. That can be used for calculation of X in specific cases. The optimum n is then determined as

$$
\hat{n} = d^2 \hat{X}^2.
$$

The nearest integer n to $\hat{n}$ is an acceptable near-optimum solution.

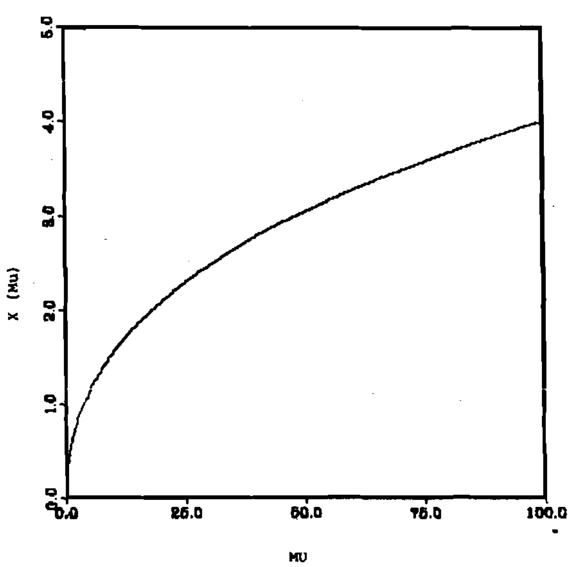

Fig. 7: $X(\mu)$ over full range of μ.

In terms of the fundamental parameters of the problem,

$$\mu = \frac{dc}{2b}\frac{1}{d^3} = \frac{c}{2bd^2}$$

$$= \frac{1}{2b}\frac{1}{\Delta}\frac{SW}{\pi V R_T}\frac{\Delta^2 R_T^2 S^2}{V^2 K^2 A}$$

$$= \frac{\Delta}{2\pi b}\frac{1}{k^2}\frac{W R_T}{A}\frac{S^3}{V^3}, \tag{79}$$

where

$$\Delta = 1 + \frac{V}{R_T}(T_{TD} + T_C). \tag{80}$$

Up to this point, we have assumed that S, the speed of the searcher, is unchanging. It is useful, however, to distinguish speed in transit to the datum, which we will call σ, from the actual search speed, S. This distinction is particularly important for helicopters which can transit rapidly to an area, but then have a low effective search speed, If we make this distinction, μ becomes

$$\mu = \frac{1}{k^2 2\pi b}\frac{S\sigma^2 W}{V^2 A}\left(R_T/V + T_{TD} + T_C\right). \tag{81}$$

Similarly

$$d^2 = \frac{k^2 A}{\sigma^2 (R_T/V + T_{TD} + T_C)^2}. \tag{82}$$

Then in the small-μ limit

$$\hat{n} \sim d^2 (0.3850)^2 \mu = 2.36 \times 10^2 \left(\frac{SW}{V^2 (R_T/V + T_{TD} + T_C)}\right) \times \left(\frac{1}{|\ln(1 - W/F)|}\right). \tag{83}$$

Conversely, in the large-μ limit

$$\hat{n} \sim d^2 \mu^{2/3} = 0.154 \left(\frac{S}{\sigma}\right)^{2/3} \left(\frac{W^{2/3} A^{1/3}}{V^{4/3} (R_T/V + T_{TD} + T_C)^{4/3}}\right) \times \left(\frac{1}{|\ln(1 - W/F)|^{2/3}}\right). \tag{84}$$

6. Asymptotic survivability of a carrier when the submarine is subjected to multiple sources of attrition

It is of some interest to understand the tactical balance between a submarine and a carrier when the submarine is subject to attrition by area forces as well as forces directly supporting the carrier. This is a situation that may arise when both the submarine and the carrier operate in a relatively confined area. Both are assumed to manoeuver at random and to encounter each other with a definable encounter rate. When an encounter occurs, three possible outcomes are interesting: (1) the submarine is sunk by the carrier's defenses; (2) the submarine sinks the carrier; or (3) the encounter ends without a definitive engagement. In the latter case, both parties continue to manoeuvre until another encounter occurs.

Between encounters with the carrier, the submarine continues to be at risk from area ASW forces. It may be attacked and destroyed at any time. The probability of that outcome can also be defined mathematically.

The issues of concern to planners are the probabilities of survival of carrier and submarine as functions of time. In an idealized case, these probabilities can be described by non-linear differential equations that can be solved in closed form. That permits one to determine, inter alia, whether the carrier has a non-zero probability of survival as a result of the action of the area ASW forces.

To formulate the problem, let $Q_S(t)$ and $Q_C(t)$ be the probabilities that the submarine and carrier are surviving at time t. They each satisfy equations of the form

$$Q_S(t+\Delta t) = Q_S(t)(1-\lambda_S \Delta t), \tag{85}$$

$$Q_C(t+\Delta t) = Q_C(t)(1-\lambda_C \Delta t), \tag{86}$$

or

$$\frac{\mathrm{d}Q_S}{\mathrm{d}t} = -Q_S(t)\lambda_S, \tag{87}$$

$$\frac{\mathrm{d}Q_C}{\mathrm{d}t} = -Q_C(t)\lambda_C. \tag{88}$$

The parameters λ_S and λ_C are the kill rates, which may in turn depend on Q_S and Q_C.

The submarine can be destroyed by area ASW forces at any time with equal likelihood. That is expressed by the use of a constant α for the rate parameter. The submarine can also be destroyed during an encounter with the carrier's defences. To express that we note that the carrier must still be surviving for the encounter to occur. Thus the rate parameter must be proportional to $Q_C(t)$. When these two effects are combined

$$\lambda_S = \alpha + \beta Q_C. \tag{89}$$

There is no independent attrition of the carrier. Thus

$$\lambda_C = \gamma Q_S, \tag{90}$$

since the submarine must survive in order to inflict damage on the carrier.

The parameters α, β, γ all have the dimensions of inverse time, or rate. α is the rate of attrition of the submarine by area forces. μ and γ are compound factors. Let μ be the rate of encounters between submarine and carrier. Then

$$\beta = \mu P_{KS/E}, \tag{91}$$

$$\gamma = \mu P_{KC/E}, \tag{92}$$

where the P's are the conditional probabilities of kill, given an encounter.

The differential equations are then

$$\frac{\mathrm{d}Q_S}{\mathrm{d}t} = -Q_S(\alpha + \beta Q_C), \tag{93}$$

$$\frac{\mathrm{d}Q_C}{\mathrm{d}t} = -Q_C(\gamma Q_S). \tag{94}$$

The initial conditions are $Q_C(0) = 1$, since the carrier is assumed to be alive at $t = 0$, and $Q_S(0) = Q_{0S}$, to allow for the possibility that the submarine may have failed to enter the operating area. For example, if the area is protected by an ASW barrier, then

$$Q_{0S} = 1 - P_{KB}, \tag{95}$$

where P_{KB} is the probability of kill at the barrier.

The governing differential equations are non-linear, and cannot be solved in closed form as a function of time.

That is not the principal issue, however. We are interested in the long term survivability of the carrier. That is, $Q_C(t)$ as $t \to \infty$. That quantity can be obtained from the equations.

Note first that the only steady-state solution of the equations

$$\frac{\mathrm{d}Q_S}{\mathrm{d}t} = \frac{\mathrm{d}Q_C}{\mathrm{d}t} = 0 \tag{96}$$

is $Q_S = 0$. Thus, the long-term behaviour implies that the probability of survival of the submarine is zero. It will be destroyed sooner or later either by the area ASW forces or by an encounter with the carrier's defenses. The question is then what limiting value does Q_C assume when $Q_S = 0$.

This can be answered by integrating the equations in the Q_C, Q_S phase plane, rather than in the time domain. When one equation is divided by the other, we have

$$\frac{\mathrm{d}Q_C}{\mathrm{d}Q_S} = \frac{\gamma Q_C}{\alpha + \beta Q_C}, \tag{97}$$

or

$$\mathrm{d}Q_C\left(\frac{\beta}{\gamma} + \frac{\alpha}{\gamma}Q_C^{-1}\right) = \mathrm{d}Q_S. \tag{98}$$

This is integrated to give

$$\frac{\beta}{\gamma}(Q_C - Q_{C0}) + \frac{\alpha}{\gamma}\ln\left(\frac{Q_C}{Q_{C0}}\right) = Q_S - Q_{0S}. \tag{99}$$

However, since $Q_{C0} = 1$ and $Q_S \to 0$, we have

$$\frac{\beta}{\gamma}(Q_C - 1) + Q_{0S} + \frac{\alpha}{\gamma}\ln Q_C = 0, \tag{100}$$

or

$$\left(\frac{\gamma Q_{0S}}{\alpha} - \frac{\beta}{\alpha}\right) + \frac{\beta}{\alpha}Q_C = -\ln Q_C. \tag{101}$$

Since $Q_C \leq 1$, both sides of this equation are positive. The value of Q_C can be found graphically.

Figure 8 is a nomogram designed for solving this equation. To use the nomogram, proceed as follows:

(1) Compute the quantity

$$A = \frac{\mu}{K}[Q_{0S}P_{KC} - P_{KS}].$$

(2) Compute the quantity

$$B = \frac{\mu}{K}[Q_{0S}P_{KC}].$$

(3) Place a straight edge on the nomogram so that it lies on the point A when $Q_{CF} = 0$ (left edge of the nomogram) and on the point B when $Q_{CF} = 1$ (right edge of the nomogram).

(4) Read on the bottom scale the value of Q_{CF} for which the straight edge intersects the curve in the nomogram. That is the desired value for carrier survivability in the face of one submarine.

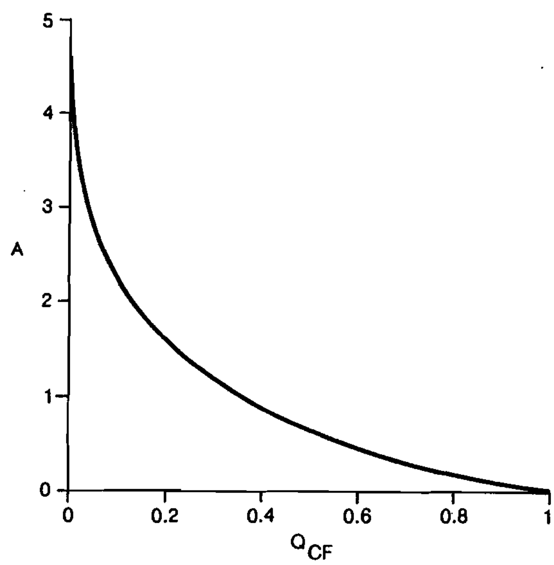

Fig. 8: **Nomogram for approximating the solution to Eq. (101).**

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